Things You Should Know Coming Into Calc I

Algebraic Rules, Properties, Formulas, Ideas and Processes:

1) **Rules and Properties of Exponents.**

Let x and y be positive real numbers, let a and b represent real numbers, and let n represent a positive integer. Then:

1) $x^a x^b = x^{a+b}$ 2) $\frac{x^a}{x^b} = x^{a-b}$

3) $(x^a)^b = x^{a \cdot b}$ 4) $(x \cdot y)^a = x^a \cdot y^a$

5) $\sqrt[n]{x} = x^{1/n}$ 6) $x^0 = 1$ so long as $x \neq 0$

7) $x^{-b} = \frac{1}{x^b}$

The sixth property is a consequence of the second property–check and see what happens if you let $a = b$. Also, the seventh property is a consequence of the second and sixth properties–check and see what happens when $a = 0$.

2) **Functions and Composition.**

A function $f(x)$ is **even** if $f(-x) = f(x)$. The graph of an even function is symmetric about the y-axis. Examples of even functions include x^2 (or indeed x^n for any even exponent n), $|x|$, and $\cos x$. Both the sum and product of a pair of even functions is also an even function.

A function $f(x)$ is **odd** if $f(-x) = -f(x)$. Note that this definition implies that if $f(x)$ is an odd function, then $f(0) = 0$. The graph of an odd function is symmetric through the origin. Examples of odd functions include x^3 (in fact, x^n for any odd exponent n) and $\sin x$. The sum of a pair of odd functions is also an odd function. The product of an even function and an odd function is an odd function. The product of two odd functions is an even function.

If $f(x)$ and $g(x)$ are a given pair of functions and if the range of $g(x)$ lies in the domain of $f(x)$, then the composition of f with g is the function defined by $(f \circ g)(x) = f(g(x))$. This is a critical and central concept to many branches of mathematics, and is one with which you should be comfortable. It is easy to see that if $e(x) = x$, then $(f \circ e)(x) = (e \circ f)(x) = f(x)$ for any x. In other words, the function $e(x)$ is an identity element with respect to functional composition. (In an analogous manner, the number 0 is the identity with respect to addition and the number 1 is the identity with respect to multiplication.) Two functions $f(x)$ and $g(x)$ are said to be composition inverses of each other (or just inverses) if $(f \circ g)(x) = (g \circ f)(x) = e(x) = x$. In this case, we commonly denote the inverse of $f(x)$ by $f^{-1}(x)$. Verify that if $f(x) = x^3$ then $f^{-1}(x) = \sqrt[3]{x}$ and if $f(x) = 4x + 7$ then $f^{-1}(x) = \frac{x-7}{4}$. Recall that one way to find the inverse of $y = f(x)$ is to write the expression $x = f(y)$ and then to solve for y.

3) **Polynomials, Factoring, and Roots.**

You should instantly realize that $x^2 - y^2 = (x + y)(x - y)$, $(x + y)^2 = x^2 + 2xy + y^2$, and $(x-y)^2 = x^2 - 2xy + y^2$. The first identity lets you rationalize the denominator of expressions such as $\frac{5}{2+\sqrt{7}}$ and $\frac{2}{4-\sqrt{x}}$. You should also be very familiar with factoring polynomials like $x^2 + 6x - 16$ and $6x^2 + 19x + 15$. Factoring a polynomial is, of course, a good way to find its roots. If a polynomial won't factor, then you can always resort to the quadratic formula. If a, b, and c are constants, then the roots of $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$. (It turns out that there is also a cubic formula and a quartic formula, but they are grotesquely complicated.) The expression b^2-4ac is called the discriminant. If the discriminant is positive, then there are two distinct real roots. If the discriminant is negative, then there are two distinct complex (or imaginary) roots. If the discriminant is 0, then there is a double root at $x = \frac{-b}{2a}$.

An important procedure we will use in the course is completing the square. If $f(x) = x^2 + bx + c$, then we complete the square by adding and subtracting $\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$ to $f(x)$. This gives us

$$f(x) = (x^2 + bx + \frac{b^2}{4}) + c - \frac{b^2}{4} = (x + \frac{b}{2})^2 + c - \frac{b^2}{4}.$$

You should be able to derive the quadratic formula by dividing both sides of $ax^2 + bx + c = 0$ by a and then completing the square.

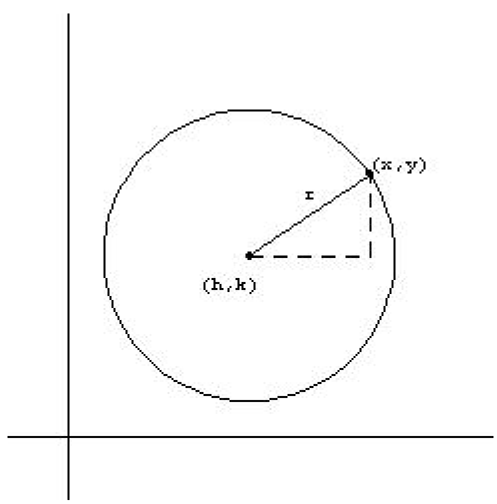
While factoring reveals the roots of a polynomial, knowing the roots can let you design a polynomial. For example, if the second degree polynomial $f(x)$ has 3 and -2 for its roots, then $f(x) = a(x+2)(x-3) = a(x^2 - x - 6)$, where some additional piece of information is needed to determine a.

4) **A Brief Review of Conic Sections.**

There are 4 conic sections, each defined by second degree polynomials. You should have some idea of what their graphs look like, and how to identify key points and features of their graphs.

A **parabola** is given by the equation $y = ax^2 + bx + c$. The parabola opens up if $a > 0$ and opens down if $a < 0$. By completing the square, you can rewrite the above equation as $y = a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a^2})$. The vertex of the parabola is at the point $(-\frac{b}{2a}, c - \frac{b^2}{4a^2})$ (basically, just keep track of the value of x that makes you square 0), and the graph of the parabola is symmetric about the vertical line $x = -\frac{b}{2a}$ that is called, amazingly enough, the axis of symmetry. Note that if $b = 0$, then the function $y = ax^2 + c$ is an even function.

A **circle** is defined to be the set of points that are a fixed distance r from a particular point (h, k) called the center of the circle. The standard form for the equation for a circle is actually nothing more than the Pythagorean Theorem! If you look at the figure, you see that $r^2 = (x - h)^2 + (y - k)^2$ where (h, k) is the center of the circle and (x, y) is some random point on the circle. Note that this example also shows that the formula for the distance between two points is also an application of the Pythagorean Theorem.



An equation of the form $ax^2 + bx + ay^2 + cy = d$ is a nonstandard equation for a circle so long as you end up with a positive number on the right hand side when, after dividing both sides by a, you complete the squares on the left hand side. If the coefficients for x^2 and y^2 are different, then the equation represents either an ellipse or, if the coefficients have opposite signs, a hyperbola. See below.

As mentioned in the above paragraph, the equation $ax^2 + bx + cy^2 + dy = e$ where $a \neq c$ is an ellipse, assuming that a and c are either both positive or both negative. (If a and c have opposite signs, then you have a hyperbola.) This equation may be rewritten as $a(x^2 + \frac{b}{a}x) + c(y^2 + \frac{d}{c}y) = e$. As usual, you can then complete the squares on the left hand side to get $a(x + \frac{b}{2a})^2 + c(y + \frac{d}{2c})^2 = e + \frac{b}{4a} + \frac{d}{4c}$. Let $E = e + \frac{b}{4a} + \frac{d}{4c}$. In order to have a real valued expression, E must be > 0 since the left hand side is a sum of two non-negative terms. Assuming a, c, and E are all positive, then so are $\frac{a}{E}$ and $\frac{c}{E}$. Thus both $\frac{E}{a}$ and $\frac{E}{c}$ are positive and could be regarded as being the squares of some pair of numbers, say $\frac{E}{a} = A^2$ and $\frac{E}{c} = B^2$. Thus the original equation for the ellipse can be put into the standard form

$$\frac{(x-h)^2}{A^2} + \frac{(y-k)^2}{B^2} = 1$$

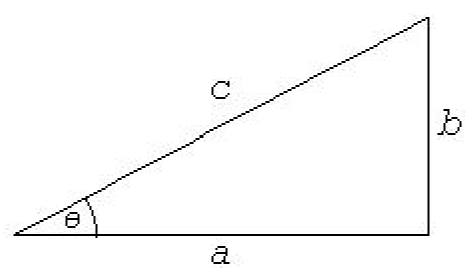
where $h = -\frac{b}{2a}$ and $k = -\frac{d}{2c}$. The center of the ellipse is at (h, k) and the vertices of the ellipse are at $(h \pm A, k)$ and $(h, k \pm B)$.

The equation $ax^2 + bx + cy^2 + dy = e$ represents a **hyperbola** when a and c have opposite signs. Proceeding as we did with the ellipse, the hyperbola can be put into one of two standard forms:

$$\frac{(x-h)^2}{A^2} - \frac{(y-k)^2}{B^2} = 1 \quad \text{or} \quad \frac{(y-k)^2}{B^2} - \frac{(x-h)^2}{A^2} = 1.$$

In the first case the graph of the hyperbola has two branches opening to the right and to the left. In the second case the hyperbola has two branches opening up and down. The center will be at (h, k) in both cases. The first hyperbola has vertices at $(h \pm A, k)$ and the second hyperbola has vertices at $(h, k \pm B)$.

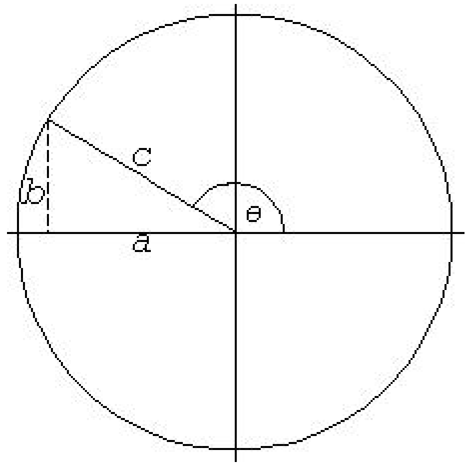
5) **A Brief Review of Trigonometry.**



Given a right triangle, you should know the definitions of the basic trig functions:

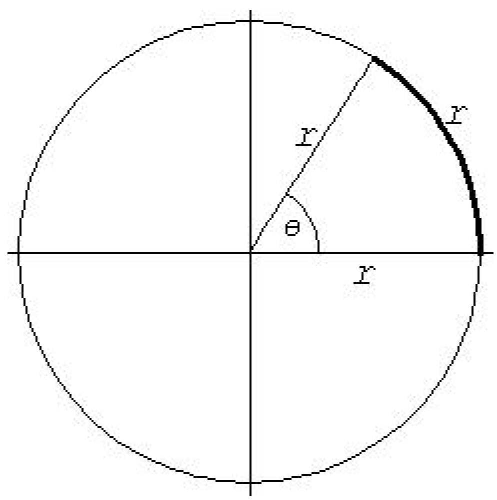
$$\sin\theta = \tfrac{b}{c} \qquad \csc\theta = \tfrac{c}{b}$$
$$\cos\theta = \tfrac{a}{c} \qquad \sec\theta = \tfrac{b}{c}$$
$$\tan\theta = \tfrac{b}{a} \qquad \cot\theta = \tfrac{a}{b}$$

The notation $\sin^2\theta$ means $(\sin\theta)^2$, which is distinct from $\sin(\theta^2)$. Note that by the Pythagorean Theorem, $\sin^2\theta + \cos^2\theta = 1$. This in turn implies that $-1 \leq \sin\theta \leq 1$ and $-1 \leq \cos\theta \leq 1$. If we base our definitions of the trig functions exclusively on triangles, then the angle θ is naturally restricted to be between 0 and 90 degrees or, in a calculus setting, between 0 and $\frac{\pi}{2}$ radians. We can extend the definitions of trig functions to arbitrary angles by using a reference triangle inscribed in a circle having its center at the origin, as in the figure below:



In this picture, the angle θ is between $\frac{\pi}{2}$ and π radians (or between 90 and 180 degrees), and so the reference triangle lies in the second quadrant. Note that a is negative since the x-coordinate of any point to the left of the y-axis is negative, and b is positive since the y-coordinate of any point above the x-axis is positive. The hypotenuse is **always** assumed to be positive.

Just as length could be measured in feet or meters, angles of rotation can be measured in degrees or radians. There are several key advantages to using radians, and they all arise from the geometric nature of the definition of a radian. Take a circle of radius r, and then mark off an arc of length r along the perimeter of the circle. (See the figure below.) The angle subtended by that arc is defined to be 1 radian. Since the circumference of a circle is $2\pi r$ and the length of the arc is r, there will be 2π such arcs in the total circumference of the circle. Hence there are 2π radians in a full circle.



Since there are 2π radians in a circle each radian represents $\frac{1}{2\pi}^{\text{th}}$ of a circle. Also, the circumference of an entire circle is $2\pi r$. Thus the length of an arc that subtends an angle of θ radians is $2\pi r \cdot \left(\frac{\theta}{2\pi}\right) = r\theta$. In a similar manner, the area of a sector of a circle that subtends an angle of θ radians is $\pi r^2 \cdot \left(\frac{\theta}{2\pi}\right) = \frac{1}{2}r^2\theta$.

SOME USEFUL IDENTITIES:

$\sin^2\theta + \cos^2\theta = 1$	$\tan^2\theta + 1 = \sec^2\theta$
$\sin(\alpha+\beta) = \sin\alpha\,\cos\beta + \cos\alpha\,\sin\beta$	$\sin 2\theta = 2\sin\theta\cos\theta$
$\cos(\alpha+\beta) = \cos\alpha\,\cos\beta - \sin\alpha\,\sin\beta$	$\cos 2\theta = \cos^2\theta - \sin^2\theta$
$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$	$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$
$\sin(-\theta) = -\sin\theta$	$\cos(-\theta) = \cos\theta$

The last two identities show that $\sin\theta$ is an odd function and $\cos\theta$ is an even function. You should have a good idea of what the graphs of both $\sin\theta$ and $\cos\theta$ look like. Also, you should know that the periods of $\sin k\theta$ and $\cos k\theta$ are $\frac{2\pi}{k}$, where k is a non-zero constant. In particular, if $k = 1$, then we see that $\sin\theta$ and $\cos\theta$ have a period of 2π.

I expect you to know the values on these tables by heart. Doing so will facilitate many of our discussions and make it easier to design manageable computational examples.

θ	$\sin\theta$	$\cos\theta$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0

Calculus I is an exciting, fun, and challenging subject. I hope that this little review gives you a nice head start on the course and will ease your transition into the course. Best of luck in the coming semester.

Dr. Thompson