The Magic of Modular Arithmetic

"Pick a Number" Games and Credit Card Dissection

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Most of us have played "pick a number" games at some point in our life - you pick a number, go through some steps, and then the book or person with whom you're playing "magically" tells you your final output.
Why do these work? What math is going on in the background?

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3. Multiply the result by 9.
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6. Multiply the result by 2.

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7. Divide the result by 3.

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9. Pick the largest (in land area) European country whose name starts with the α^{th} letter of the alphabet.

1. Austria
2. Belarus
3. Czech Republic
4. Denmark
5. Estonia
6. France
7. Germany
8. Hungary
9. Italy
10. (None start with J)
11. (None start with K)
12. Lithuania
13. Moldova
14. Norway
15. (None start with O)
16. Poland
17. (None start with Q)
18. Romania (unless you count Russia)
19. Spain
20. (None start with T)
21. Ukraine
22. Vatican City
23. (None start with W)
24. (None start with X)
25. (None start with Y)
26. (None start with Z)

10. Choose a color whose first letter is the last letter of the country you chose in the previous step.

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11. You chose yellow.

Step 1 - "Pick a number from 1 to 9."

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Call it "n". Clearly, $1 \leq n \leq 9$.

Step 2 - "Add it to 4."

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Now we have $n + 4$, and so $5 \leq n + 5 \leq 13$.

Step 3 - "Multiply the result by 9."

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This gives us:

$$9(n + 4) = 9n + 36$$

Step 4 - "Add 6 to the result."

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(We can think of this either way, but $9n + 36 + 6$ is a little easier to work with in the next step.)

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An integer a is divisible by 9 if and only if 9 divides the sum of the digits of a.

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Example

38 and 2 are congruent modulo 9 since $38/9 = 4 \text{ r } 2$ and $2/9 = 0 \text{ r } 2$

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Since our initial choice of n was from 1 to 9, the sum of the digits of the number $9n$ will be 9 (by the theorem), and the sum of the digits of the number $9n + 42$ will be less than or equal to

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The only two numbers satisfying both of these conditions are 6 and 15.

Step 6 - "Multiply the result by 2."

Since we know the result of step 5 is either 6 or 15, multiplying by 2 will yield 12 or 30.

Step 7 - "Divide the result by 3."

This step leaves us with either 4 or 10.

Step 8 - "Add 4 to the result to get a number - call it α"

Now we have 8 or 14.

Step 9 - "Pick the largest (in land area) European country whose name starts with the α^{th} letter of the alphabet."

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Step 9 - "Pick the largest (in land area) European country whose name starts with the α^{th} letter of the alphabet."

With any "magical pick-a-number" problem, we eventually need to get to the point where the person playing the game is forced into making a particular choice. At this point in our game, the player will have chosen either Hungary or Norway. We haven't narrowed it down to one possibility yet, but since these words share the same last letter we can force a choice in the next step.

Step 10 - "Choose a color whose first letter is the last letter of the country you chose in the previous step."

Both of countries that the player could have chosen in the last step end in "y", so we're finally down to one choice.

Step 11 - "You chose yellow."

Even if the player is trying to be difficult, there aren't really any options other than yellow :-)

Thank you!

References

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- http://www.nationsencyclopedia.com/Europe/index.html