MATH 213: Logical Equivalences, Rules of Inference and Examples

Tables of Logical Equivalences

Note: In this handout the symbol $\equiv$ is used the tables instead of $\iff$ to help clarify where one statement ends and the other begins, particularly in those that have a biconditional as part of the statement. The abbreviations are not universal.

Equivalence	**Name**	**Abbr.**
$p \wedge \mathrm{T} \equiv p$	Identity / Idempotent (Conjunction)	IdC
$p \vee \mathrm{F} \equiv p$	Identity / Idempotent (Disjunction)	IdD
$p \wedge \mathrm{F} \equiv \mathrm{F}$	Domination (Conjunction)	DomC
$p \vee \mathrm{T} \equiv \mathrm{T}$	Domination (Disjunction)	DomD
$\neg(\neg p) \equiv p$	Double Negation	DN
$p \wedge q \equiv q \wedge p$	Commutative (Conjunction)	CC
$p \vee q \equiv q \vee p$	Commutative (Disjunction)	CD
$(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative (Conjunction)	AC
$(p \vee q) \vee r \equiv p \vee (q \vee r)$	Associative (Disjunction)	AD
$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive (Conjunction)	DC
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$	Distributive (Disjunction)	DD
$\neg(p \wedge q) \equiv \neg p \vee \neg q$	DeMorgan's Law (Conjunction)	DMC
$\neg(p \vee q) \equiv \neg p \wedge \neg q$	DeMorgan's Law (Disjunction)	DMD
$p \wedge (p \vee q) \equiv p$	Absorption (Conjunction)	AbC
$p \vee (p \wedge q) \equiv p$	Absorption (Disjunction)	AbD
$p \wedge \neg p \equiv \mathrm{F}$	Negation (Conjunction)	NegC
$p \vee \neg p \equiv \mathrm{T}$	Negation (Disjunction)	NegD

Table 1: Logical Equivalences

Equivalence	Name	Abbr
$\neg(p \implies q) \equiv p \wedge \neg q$	Negation of Implication	NI
$p \implies q \equiv \neg p \vee q$	Implication to Disjunction	ID
$p \implies q \equiv \neg q \implies \neg p$	Contrapositive	C
$p \vee q \equiv \neg p \implies q$		
$p \wedge q \equiv \neg(p \implies \neg q)$		
$(p \implies q) \wedge (p \implies r) \equiv p \implies (q \wedge r)$		
$(p \implies r) \wedge (q \implies r) \equiv (p \vee q) \implies r$		
$(p \implies q) \vee (p \implies r) \equiv p \implies (q \vee r)$		
$(p \implies r) \vee (q \implies r) \equiv (p \wedge q) \implies r$		

Table 2: Logical Equivalences Involving Implications

Equivalence	Name	Abbr.
$\neg(p \iff q) \equiv \neg p \iff q$	Negation of Biconditional	NB
$\neg(p \iff q) \equiv p \iff \neg q$	Negation of Biconditional (alternative)	NB
$p \iff q \equiv (p \implies q) \wedge (q \implies p)$	Biconditional	B
$p \iff q \equiv \neg p \iff \neg q$	Contrapositive of Biconditional	
$p \iff q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$		

Table 3: Logical Equivalences Involving Biconditionals

Tautology (so these will be true for an	Name	Abbr.
$p \vee \neg p$	Excluded Middle	EM
$(p \wedge q) \implies p$	Simplification	S
$p \implies (p \vee q)$	Addition	A
$(p \wedge (p \implies q)) \implies q$	Modus Ponens	MP
$((p \implies q) \wedge (q \implies r)) \implies (p \implies r)$	Hypothetical Syllogism	HS
$((p \vee q) \wedge \neg q) \implies p$	Disjunctive Syllogism	DS
$(\neg q \wedge (p \implies q)) \implies \neg p$	Modus Tollens	MT
$((p \vee r) \wedge ((p \implies q) \wedge (r \implies s))) \implies (q \vee s)$	Constructive Dilemma	CDL
$((\neg q \vee \neg s) \wedge ((p \implies q) \wedge (r \implies s))) \implies (\neg p \vee \neg r)$	Destructive Dilemma	DDL
$(p \vee p) \implies p$	Idempotent	IM

Table 4: Additional Tautologies

(Remember, *tautology* means these will always be true for any values of p, q, r, and s.)

Standard Rules of Inference

Each of the following is based on a tautology.

- Modus Ponens $\begin{array}{rl} & p \\ & p \implies q \\ \hline \therefore & q \end{array}$

- Modus Tollens $\begin{array}{rl} & \neg q \\ & p \implies q \\ \hline \therefore & \neg p \end{array}$

- Conjunctive Simplification $\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \end{array}$

- Disjunctive Syllogism $\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$

- Hypothetical Syllogism $\begin{array}{rl} & p \implies q \\ & q \implies r \\ \hline \therefore & p \implies r \end{array}$

. . . .

Others not give in the book:

- Addition $\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$

- Conjunctive Simplification (alternate version) $\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$

- Resolution $\begin{array}{rl} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$

Example 1. Identify the rules of inference used in each of the following arguments.

(a) Alice is a math major. Therefore, Alice is either a math major or a c.s. major.

(b) If it snows today, the college will close. The college is not closed today. Therefore it did not snow today.

(c) If I go swimming, then I will stay in the sun too long. If I stay in the sun too long, then I will sunburn. Therefore, if I go swimming, then I will get sunburn.

Example 2. Use rule of inference to show that the premises "Henry works hard", "If Henry works hard then he is a dull boy", and "If Henry is a dull boy then he will not get the job" imply the conclusion "Henry will not get the job."

Standard Rules of Inference

Each of the following is based on a tautology.

- Universal Instantiation
$$\begin{array}{rl} & \forall x\, P(x) \\ \hline \therefore & P(c) \text{ for any fixed } c \end{array}$$

- Universal Generalization
$$\begin{array}{rl} & P(c) \text{ for an arbitrary } c \\ \hline \therefore & \forall x\, P(x) \end{array}$$

- Existential Instantiation
$$\begin{array}{rl} & \exists x\, P(x) \\ \hline \therefore & P(c) \text{ for some } c \end{array}$$

- Existential Generalization
$$\begin{array}{rl} & P(c) \text{ for some } c \\ \hline \therefore & \exists x\, P(x) \end{array}$$

- Universal Modus Ponens
$$\begin{array}{rl} & \forall x\, (P(x) \implies Q(x)) \\ & P(c) \\ \hline \therefore & Q(c) \end{array}$$

- Universal Modus Tollens
$$\begin{array}{rl} & \forall x\, (P(x) \implies Q(x)) \\ & \neg Q(c) \\ \hline \therefore & \neg P(c) \end{array}$$

Example 3. What can you conclude about Henry, Jack, and Jill, given the following premises?

1. Every c.s. major has an iPad.

2. Henry does not have an iPad.

3. Jill has an iPad.

4. Jack is a c.s. major.

Fallacies

- Affirming the Conclusion $$\begin{array}{rl} & p \implies q \\ & q \\ \hline \therefore & p \end{array}$$

- Universal A.C. $$\begin{array}{rl} & \forall x \in S,\ P(x) \implies Q(x) \\ & \text{For a particular } s \in S,\ Q(s). \\ \hline \therefore & P(s). \end{array}$$

- Denying the Hypothesis $$\begin{array}{rl} & p \implies q \\ & \neg p \\ \hline \therefore & \neg q \end{array}$$

- Universal D.H. $$\begin{array}{rl} & \forall x \in S,\ P(x) \implies Q(x) \\ & \text{For a particular } s \in S,\ \not P(s). \\ \hline \therefore & \neg Q(s). \end{array}$$

Example 4 (Valid or Fallacy?)**.** Do the following represent valid arguments, or fallacies?

(a) All students in this class understand logic. Pascal is a student in this class. Therefore, Pascal understands logic. (Let $P(x)$ = "x is in this class" and $Q(x)$ = "x understands logic".)

(b) Every c.s. major takes discrete mathematics. Esther is taking discrete mathematics. Therefore, Esther is a c.s. major. (Let $P(x)$ = "x is a c.s. major" and $Q(x)$ = "x takes discrete".)

(c) All parrots like fruit. My pet bird is not a parrot. Therefore, my pet bird does not like fruit. (Let $P(x)$ = "x is a parrot" and $Q(x)$ = "x like fruit".)

(d) Everyone who eats granola every day is healthy. John is not healthy. Therefore John does not eat granola every day. (Let $P(x)$ = "x eats granola every day" and $Q(x)$ = "x is healthy".)

Example 5. (If we can find the link.) Let

- $F(x)$ ="x's cable freezes"
- $I(x)$ ="x is irritable"
- $S(x)$ ="x's work suffers"
- $W(x)$ ="x is wrongly convicted"
- $T(x, y)$ ="x has time to think about y (a lot)"
- $E(x)$ ="x's house explodes"
- $D(x)$ ="x has DirecTV®"

Additional Puzzles

1. Use all of the following premises to reach a conclusion:
   - The only books in this library, that I do not recommend for reading, are unhealthy in tone.
   - The bound books are all well written.
   - All the romances are healthy in tone.
   - I do not recommend you to read any of the unbound books.
2. [1]Use all of the following premises to reach a conclusion:
   - All my sons are slim.
   - No child of mine is healthy who takes no exercise.
   - All gluttons, who are children of mine, are fat.

[1]Puzzles 1 and 2 are attributed to Lewis Carroll

- No daughter of mine takes any exercise.

3. **The Lady or the Tiger.**[2] A certain king likes to entertain himself by making his prisoners play a game to decide their fate. The prisoners are presented with two doors. In a room behind each door is either a lady whom the prisoner may marry, or a tiger whom may eat the prisoner. A clue is written on each door and the prisoner decides which door to open based on these clues. The clues provided to three prisoners brought before the king are below. Try to figure out which door each prisoner should open.

   Prisoner 1 is told that exactly one of the following clues is true and exactly one is false.

   Door 1: There is a lady behind this door and a tiger behind the other.

   Door 2: There is a lady behind one of the doors and a tiger behind the other.

   Prisoner 2 is told that either both clues are true or both are false.

   Door 1: Either there is a tiger behind this door or a lady behind the second door.

   Door 2: There is a lady behind this door.

   Prisoner 3 receives directions which are a bit tricker since the first two escaped. This prisoner is told that if a lady is behind door 1 then the clue on door 1 is true, but if a tiger is behind door 1 then the clue on that door is false. Door 2 follows the opposite rule: if a lady is behind door 2 the clue on door 2 is false, but if a tiger is behind door 2 the clue on that door is true.

   Door 1: A lady is waiting behind at least one of the doors.

   Door 2: A lady is waiting behind the other door.

4. Let us assume that there are five houses of different colors next to each other on the same road. In each house lives a man of a different nationality. Every man has his favorite drink, his favorite brand of cigarettes, and keeps pets of a particular kind.

   - The Englishman lives in the red house.
   - The Swede keeps dogs.
   - The Dane drinks tea.
   - The green house is just to the left of the white one.
   - The owner of the green house drinks coffee.
   - The Pall Mall smoker keeps birds.
   - The owner of the yellow house smokes Dunhills.
   - The man in the center house drinks milk.
   - The Norwegian lives in the first house.
   - The Blend smoker has a neighbor who keeps cats.
   - The man who smokes Blue Masters drinks bier.
   - The man who keeps horses lives next to the Dunhill smoker.
   - The German smokes Prince.
   - The Norwegian lives next to the blue house.
   - The Blend smoker has a neighbor who drinks water.

   Who keeps fish as his pet?[3]

[2]Commonly attributed to Raymond Smullyan.
[3]Commonly attributed to Albert Einstein.

Symbolic Proofs using Rules of Inference

Example 6. Give an argument (based on rules of inference) to show that the hypotheses/premises

$$(\neg p \wedge q) \implies (r \vee s), \quad \neg p \implies (r \implies w), \quad (s \implies t) \vee p, \quad \neg p \wedge q$$

lead to the conclusion $w \vee t$.

Line	**Step**	**Reason**
(1)		
(2)		
(3)		
(4)		
(5)		

Example 7. Give an argument (based on rules of inference) to show that the hypotheses/premises

$$p \implies q, \quad \neg q \vee r, \quad r \implies (t \vee s), \quad \neg s \wedge p$$

lead to the conclusion t.

Line	**Step**	**Reason**
(1)		
(2)		
(3)		
(4)		
(5)		

Example 8. Give an argument (based on rules of inference) to show that the hypotheses/premises

$$p \wedge q,\ p \implies (\neg q \vee r),\ r \implies s$$

lead to the conclusion s.

Line	**Step**	**Reason**
(1)		
(2)		
(3)		
(4)		
(5)		