

Area Between Curves Volumes of Solids of Revolution

Area Between Curves

Theorem: Let $f(x)$ and $g(x)$ be continuous functions on the interval $[a, b]$ such that $f(x) \geq g(x)$ for all x in $[a, b]$. Then the area of the region between $f(x)$ and $g(x)$ on $[a, b]$ is

$$\int_a^b f(x) - g(x) dx$$

or, less formally,

$$\int_a^b \text{upper} - \text{lower} dx \quad \left(\text{or } \int_c^d \text{right} - \text{left} dy \right)$$

Steps:

To find the area of the region between two curves $f(x)$ and $g(x)$:

1. Set the two functions equal and solve for x to find any intersections points. *Note:* We'll do this even if we're given an interval in the problem (the functions could cross at some point in the interval, changing which is the upper and which is the lower function) *unless* we're explicitly told/shown that they don't cross in that interval.

If we are given an interval, then we only need the intersection points that lie in that interval.

2. We set up one integral for each pair of adjacent intersection/end points.

- (a) If we're trying to find the area between f and g over a given interval $[a, b]$ and the functions intersect at x_1, x_2, \dots, x_n **in** $[a, b]$, then we would have

$$\int_a^{x_1} (\text{upper}_1 - \text{lower}_1) dx, \int_{x_1}^{x_2} (\text{upper}_2 - \text{lower}_2) dx, \dots, \int_{x_n}^b (\text{upper}_{n+1} - \text{lower}_{n+1}) dx$$

- (b) If we're trying to find the area between f and g overall (no interval given) and the functions intersect at x_1, x_2, \dots, x_m , then we would have

$$\int_{x_1}^{x_2} (\text{upper}_1 - \text{lower}_1) dx, \int_{x_2}^{x_3} (\text{upper}_2 - \text{lower}_2) dx, \dots, \int_{x_{m-1}}^{x_m} (\text{upper}_{m-1} - \text{lower}_{m-1}) dx$$

Adding up these integrals gives us the total area bounded by the two curves (over the interval, if given).

3. For any of these integrals, if we subtract the functions in the wrong order inside the integral, then the answer will change sign (negative instead of positive). If that were to happen, we could simply take the absolute value of that number to find the correct answer.

To save ourselves some time and the trouble of having to either graph the functions or pick test values to see which is the upper and which is the lower over each of those intervals, we'll simply subtract them in the same order in all of the integrals, evaluate, and then take the absolute value of each **before** adding them together to get the total area. For example, in (b) above, the total area would be

$$\left| \int_{x_1}^{x_2} f(x) - g(x) dx \right| + \left| \int_{x_2}^{x_3} f(x) - g(x) dx \right| + \dots + \left| \int_{x_{m-1}}^{x_m} f(x) - g(x) dx \right|$$

Examples

1. Let

$$f_1(x) = 2 - x^2, \quad f_2(x) = x^2, \quad f_3(x) = 1 - x^2, \quad f_4(x) = x^2 - 1, \quad f_5(x) = -x^2, \quad f_6(x) = x^2 - 2$$

(a) Find the area between f_1 and f_2 .

(b) Find the area between f_3 and f_4 .

(c) Find the area between f_5 and f_6 .

(d) Find the area between f_1 and f_2 over the interval $[-\frac{1}{2}, \frac{1}{2}]$.

(e) Find the area between f_1 and f_2 over the interval $[0, 2]$.

2. Let

$$y_1 = 2x^2, \quad y_2 = x^3 - 3x.$$

(a) Find the area between y_1 and y_2 .

(b) Find the area between y_1 and y_2 over the interval $[-2, 3]$.

Volumes of Solids of Revolution

Summary of Methods:

The point a (or c) is where the slices/region begins and the point b (or d) is where the slices/region ends.

Disk Method:

$$V = \int_a^b \pi [R(x)]^2 dx$$

$$\text{(or } V = \int_c^d \pi [R(y)]^2 dy)$$

Washer Method:

$$V = \int_a^b \pi \left([R(x)]^2 - [r(x)]^2 \right) dx$$

$$\text{(or } V = \int_c^d \pi \left([R(y)]^2 - [r(y)]^2 \right) dy)$$

Shell Method:

$$V = \int_a^b 2\pi r(x) h(x) dx$$

$$\text{(or } V = \int_c^d 2\pi r(y) h(y) dy).$$

Note: When your slices/shells are parallel to the x -axis, everything in the integral should be in terms of x (including the limits of integration); when your slices/shells are parallel to the y -axis, everything in the integral should be in terms of y (including the limits of integration).

Tips to Remember:

1. *Always* sketch the region and an outline of the solid first!! You need enough detail and accuracy to be able to decide on a method and set up the integral.
2. As mentioned above, your slices/shells will always be perpendicular to the axis corresponding to the variable you are integrating with respect to (x or y).
 - (a) If you are rotating about the line $y = \underline{\hspace{1cm}}$ (remember that the x -axis is the line $y = 0$), then [in this class] you will have to use either the disk or washer method (depending on the region) and integrate wrt x . (This is because we're not using shells wrt y).
 - (b) If you are rotating about the line $x = \underline{\hspace{1cm}}$ (remember that the y -axis is the line $x = 0$), then [in this class] the shell method will work (integrate wrt x), but you may also be able to use disks or washers and integrate wrt y depending on the function(s) (and this may be the easier way to do the problem).
3. Set up the limits of integration based on the **original** two-dimensional region, *not* where the rotated solid lies. (It may help to shade the original region so you don't confuse it with the solid.) Your radius (and height, if applicable) function(s) will need to give the correct values based on x 's (or y 's) within those limits.
4. When trying to determine the radius, r or R , you need to find the distance from the outside of the shape to the axis of rotation. Think of this as the top curve minus the bottom curve, or the right curve minus the left curve. You should always get a positive value (since it is a distance), so you can double check by plugging in an x or y value that is in the region.

5. Check that your functions do what you want them to do. It's usually enough to plug in the upper and lower limits of integration to see if they agree.
6. When using the shell method, your radius function will always be a linear function of the form $r(x) = \pm x + C$ or $r(y) = \pm y + C$, where C is some real number (so C can be negative or 0).