1 Limit Laws

Suppose $c$ is constant, $n$ is a positive integer, and $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist. Then,

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [c \cdot f(x)] = c \cdot \left[ \lim_{x \to a} f(x) \right]$
4. $\lim_{x \to a} [f(x) \cdot g(x)] = \left[ \lim_{x \to a} f(x) \right] \cdot \left[ \lim_{x \to a} g(x) \right]$
5. $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, as long as $\lim_{x \to a} g(x) \neq 0$.
6. $\lim_{x \to a} [f(x)]^n = \left[ \lim_{x \to a} f(x) \right]^n$
7. $\lim_{x \to a} c = c$
8. $\lim_{x \to a} x = a$
9. $\lim_{x \to a} x^n = a^n$
10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$. If $n$ is even, then only true for $a > 0$.
11. $\lim_{x \to a} \sqrt{f(x)} = \sqrt{\lim_{x \to a} f(x)}$. If $n$ is even, then only true when $\lim_{x \to a} f(x) > 0$.

These laws lead to the following:

1.1 Direct Substitution Property

If $f(x)$ is a polynomial [remember that this includes constant and linear functions], rational, radical, or trigonometric function and $a$ is in the domain of $f$ (and not a zero of $f$ if $f$ is a radical function), then

$$\lim_{x \to a} f(x) = f(a)$$

(i.e., we can evaluate the limit by simply plugging in $x = a$).

In general, direct substitution should work as long as nothing “goes wrong” at or around $x = a$. Examples of something “going wrong” would be vertical asymptotes, a function being undefined on one side of $a$, or changing to another piece of a piecewise-defined function.
2 Algebraic Techniques for Finding Limits

First a useful theorem:

**Theorem 1.** If \( f(x) = g(x) \) except at \( x = a \), then \( \lim_{x \to a} f(x) = \lim_{x \to a} g(x) \) (if the limits exist).

This theorem makes some of the algebraic manipulations we’ll be doing “legal”.

2.1 Algebraic Techniques

Here are some common algebraic techniques that arise when finding limits:

- Canceling factors. (This is possible because of the theorem above.)
- Putting terms over a common denominator.
- Rationalizing, i.e., multiplying/dividing by a conjugate.
- Analyzing behavior under a radical.
- A combination of the above.

2.2 Finding Limits of \( \pm \infty \)

If, after canceling as much as possible, we have \( f(a) = \frac{c}{0} \), where \( c \neq 0 \), then

\[
\lim_{x \to a \pm} f(x) = \pm \infty
\]

(As long as \( f(x) \) is a “nice” function, such as a rational or trig function.)

To determine whether the limit is \( +\infty \) or \( -\infty \), we need to determine whether the function is positive or negative near \( a \) (typically we’ll need to check on the left and right of \( a \) separately).

2.3 Limits Involving Absolute Value

Recall that a function involving absolute value can be expressed as a piecewise-defined function. For example,

\[
|x| = \begin{cases} 
  x, & \text{if } x \geq 0 \\
  -x, & \text{if } x < 0 
\end{cases}
\]

When evaluating the limit of a function involving absolute value, we first write the function in piecewise form, then take left- and right-hand limits (using whichever technique is appropriate).

2.4 Squeeze Theorem

**Squeeze Theorem.** If

\[
f(x) \leq g(x) \leq h(x)
\]

when \( x \) is near \( a \), but not necessarily at \( a \) [for instance, \( g(a) \) may be undefined], and \( \lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L \), then \( \lim_{x \to a} g(x) = L \) also.
2.4.1 Examples

Example 1. Find
\[ \lim_{x \to 0} x^2 \cos \left( \frac{1}{x^2} \right). \]

When trying to find functions to use to ‘squeeze’ \( g(x) \), we want functions that are, a) similar enough to \( g(x) \) that we can be sure the squeeze works, b) easier to evaluate their limit as \( x \to a \). We typically do this by starting with the most complicated or troublesome part of \( g(x) \), see if we can find constants (or simpler functions) that it stays between, and then multiply in the rest of ‘nicer’ parts of \( g(x) \).

In this case, the part of \( g(x) \) that is giving us the most trouble is the \( \cos \left( \frac{1}{x^2} \right) \) part (we get division by 0 if we try direct substitution). Now we know that cosine stays between -1 and 1, so

\[ -1 \leq \cos \left( \frac{1}{x^2} \right) \leq 1 \]

for any \( x \) in the domain of the function (i.e., any \( x \neq 0 \)). Since \( x^2 \) is always positive, we can multiply this inequality through by \( x^2 \):

\[ -x^2 \leq x^2 \cos \left( \frac{1}{x^2} \right) \leq x^2 \]

So, our original function is bounded by \( -x^2 \) and \( x^2 \). Now since

\[ \lim_{x \to 0} -x^2 = \lim_{x \to 0} x^2 = 0, \]

then, by the Squeeze Theorem,

\[ \lim_{x \to 0} x^2 \cos \left( \frac{1}{x^2} \right) = 0. \]

Example 2. Find
\[ \lim_{x \to 0} x^2 e^{\sin \left( \frac{1}{x} \right)}. \]

As in the last example, we have a problem arising from division by 0 inside the trig term. Now the range of sine is also \([-1, 1]\), so

\[ -1 \leq \sin \left( \frac{1}{x} \right) \leq 1. \]

Taking \( e \) raised to both sides of an inequality does not change the inequality, so

\[ e^{-1} \leq e^{\sin \left( \frac{1}{x} \right)} \leq e^1, \]

and, as in the last example, we can multiply through by \( x^2 \) and get

\[ x^2 e^{-1} \leq x^2 e^{\sin \left( \frac{1}{x} \right)} \leq x^2 e^1. \]

So, our original function is bounded by \( e^{-1}x^2 \) and \( ex^2 \), and since

\[ \lim_{x \to 0} e^{-1}x^2 = \lim_{x \to 0} ex^2 = 0, \]

then, by the Squeeze Theorem,

\[ \lim_{x \to 0} x^2 e^{\sin \left( \frac{1}{x} \right)} = 0. \]
2.5 Two Special Limits

\[ \lim_{x \to 0} \frac{\sin(x)}{x} = 1, \quad \lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0 \]

Example 3. Find

\[ \lim_{x \to 0} \frac{\sec(x) - 1}{x \sec(x)} \]

Direct substitution leads to division by 0. If we rewrite the secants as \( \frac{1}{\cos(x)} \)’s and do some rearranging, we can write this as limit as:

\[
\begin{align*}
\lim_{x \to 0} \frac{\sec(x) - 1}{x \sec(x)} &= \lim_{x \to 0} \frac{1}{x} \left( \frac{1}{\cos(x)} - 1 \right) \\
&= \lim_{x \to 0} \frac{1 - \cos(x)}{x \cos(x)} \\
&= - \left( \lim_{x \to 0} \frac{\cos(x) - 1}{x} \right) \\
&= - (0) = 0
\end{align*}
\]

(We could have also multiplied and divided the original equation by \( \cos(x) \) and simplified. The limit would been the same.)

2.6 Finding Limits at \( \pm \infty \)

Definition 1.

\[ " \lim_{x \to \infty} f(x) = L" \]

means that we can make \( f(x) \) arbitrarily close to \( L \) by taking sufficiently large positive values of \( x \). So, as \( x \) grows larger, the function values approach \( L \).

Similarly,

\[ " \lim_{x \to -\infty} f(x) = L" \]

means that we can make \( f(x) \) arbitrarily close to \( L \) by taking sufficiently large negative values\(^1\) of \( x \). So, as \( x \) grows larger in the negative direction, the function values approaches \( L \).

The line \( y = L \) is called a horizontal asymptote.

We can also combine limits at \( \pm \infty \) with the idea of limits of \( \pm \infty \) and talk about

\[ \lim_{x \to \infty \text{ or } x \to -\infty} f(x) = \infty \text{ or } -\infty \]

\(^1\)Talking about a ‘large negative number’ is somewhat ambiguous, but we will take it to mean that the number is negative and far away from 0, i.e., the number is negative and its magnitude (absolute value) is large.
The following theorem is used when evaluating almost all infinite limits:

**Theorem 2.** Let $c$ be a real number and let $r$ be a positive rational number. Then

$$
\lim_{x \to \infty} \frac{c}{x^r} = 0
$$

If $x^r$ is defined for all values of $x$ in the interval $(-\infty, 0)$ then

$$
\lim_{x \to -\infty} \frac{c}{x^r} = 0
$$

When taking the limit as $x \rightarrow -\infty$ we can’t have a value of $r$ that is equivalent to taking an even root (e.g., $r = \frac{1}{2}$ would cause us to be taking the square root of negative values).

To evaluate limits at $\pm\infty$ we will almost always divide through by the largest power of $x$ in the denominator, simplify, and then use the result of the theorem to evaluate.

While we can’t really “plug in” $x = \pm\infty$ (since $\infty$ isn’t a number), we can sometimes think that way and use the following sloppy notation to evaluate certain limits at infinity. (Note: in all of the following, when we write “$\infty$” what we really mean is “something approaching $\infty$ [as $x$ approaches some value]”; similarly, “$-\infty$” really means “something approaching $-\infty$ [as $x$ approaches some value]”.)

- $\infty \cdot c = \begin{cases} 
\infty, & \text{if } c > 0 \\
0, & \text{if } c = 0 \\
-\infty, & \text{if } c < 0
\end{cases}$ (This is *not* the same as $c$ approaching 0.)
- $-\infty \cdot c = \begin{cases} 
-\infty, & \text{if } c > 0 \\
0, & \text{if } c = 0 \\
\infty, & \text{if } c < 0
\end{cases}$ (Again, this is *not* the same as $c$ approaching 0.)
- $\infty \cdot \infty = \infty$
- $-\infty \cdot -\infty = \infty$
- $-\infty \cdot \infty = -\infty$
- $\infty \cdot -\infty = -\infty$
- $\infty + \infty = \infty$
- $-\infty - \infty = -\infty$
- $\infty + c = \infty$ for any real number $c$.
- $-\infty + c = -\infty$ for any real number $c$.
- $\infty - \infty = ??$ We cannot say anything about this form!
- $\pm \infty \cdot ($something approaching 0$) = ??$ We cannot say anything about this form, either!