Chapter 5 - Induction

5.1 - Mathematical Induction

Axiom 1 (The Principle of Mathematical Induction (PMI)). To prove that the predicate $P(n)$ is true for all $n \in A$, where $A \subseteq \mathbb{N}$, we must do the following:

Basis Step: Show that $P(n_0)$ is true, where $n_0$ is the smallest $n$ value for which $P(n)$ is true.

Inductive Step: Show that for all $k \geq n_0$, if $P(k)$ is true then $P(k+1)$ is also true. (Equivalently, you can show that for all $k > n_0$, if $P(k-1)$ is true then $P(k)$ is also true.)

Metaphors: Climb a ladder. Eat an elephant.

Template for Induction

Here is a more detailed outline of the underlying structure of an induction proof:

Basis step / Base case: Show that $P(n_0)$ is true, where $n_0$ is the smallest $n$ value for which $P(n)$ is true.

Induction hypothesis: Assume that $P(k)$ is true for $k \geq n_0$.

Induction step: Now show that $P(k) \rightarrow P(k+1)$ (alternatively, show that $P(k-1) \rightarrow P(k)$). In other words we must show that $P(k+1)$ is true by using the fact that $P(k)$ is true. For instance, if we are proving an equality, start with one side of the equality and show through a series of steps (including using the equality given by $P(k)$) that we can get to the other side of the equality.

Conclusion: By induction (or by PMI), $P(n)$ is true for all integers $n \geq n_0$. 

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Example 1. Prove that for all $n \in \mathbb{Z}_+$, \[ \sum_{i=1}^{n} (2i - 1) = n^2. \]
Example 2. Prove that for all $n \in \mathbb{Z}_+$, \[
\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}.
\]
Example 3. A group of people stand in line to purchase movie tickets. The first person in line is a woman and the last person in line is a man. Use proof by induction to show that somewhere in the line a woman is directly in front of a man.
5.2 - Strong Induction

**Axiom 2** (Strong Mathematical Induction (2PMI)). *To prove that the predicate \( P(n) \) is true for all \( n \in A \), where \( A \subseteq \mathbb{N} \), we must do the following:*

**Basis Step:** Show that \( P(n_0) \) is true, where \( n_0 \) is the smallest \( n \) value for which \( P(n) \) is true.

**Inductive Step:** Show that for all \( k \geq n_0 \), if \( P(n_0), P(n_0 + 1), P(n_0 + 2), \ldots, P(k) \) are all true then \( P(k + 1) \) is also true. (Equivalently, show that if \( P(n_0), P(n_0 + 1), P(n_0 + 2), \ldots, P(k - 1) \) are all true then \( P(k) \) is also true.)

**Template for Strong Induction**

Here is a more detailed outline of the underlying structure of an induction proof:

**Basis steps / Base cases:** Show that \( P(n_0) \) and \( P(n_0 + 1) \) and, \ldots, and \( P(n_0 + j) \) are all true where \( n_0 \) is the smallest \( n \) value for which \( P(n) \) is true and \( j \) is the number of bases cases needed so that we never use anything before \( P(n_0) \) in the induction step.

**Induction hypothesis:** Let \( k \geq n_0 + j \) and assume that \( P(n_0) \) and \( P(n_0 + 1) \) and, \ldots, and \( P(k) \) are all true (alternatively, assume up through \( P(k - 1) \) are true).

**Induction step:** Now show that \( P(n_0) \land P(n_0 + 1) \land \cdots \land P(k) \rightarrow P(k + 1) \) (alternatively, show that \( P(n_0) \land P(n_0 + 1) \land \cdots \land P(k - 1) \rightarrow P(k) \)). In other words we must show that \( P(k + 1) \) is true by using the fact that \( P(n_0) \) and \( P(n_1) \) and \ldots and \( P(k) \) are all true. (Usually this will involve simplifying the \( k + 1 \) case so that it is equal to two or more smaller cases for which we know the induction hypothesis holds.)

**Conclusion:** By strong induction (or by 2PMI), \( P(n) \) is true for all integers \( n \geq n_0 \).

**Example 4.** Show that any amount of postage totaling 50¢ or more can be composed using 6¢ and 11¢ stamps.
Example 5. Prove that any integer greater than 1 is either prime or the product of two or more primes.
Example 6. Assume that a chocolate bar consists of $n$ squares arranged in a rectangular pattern. The bar, or a smaller rectangular piece of the bar, can be broken along a vertical or horizontal line separating the squares. Assuming that only one break can be made at a time and that each break separates a rectangular bar into two smaller rectangular bars, prove that it takes $n - 1$ successive breaks to break the bar into $n$ separate squares.
Example 7. Prove that \( \frac{d(x^n)}{dx} = nx^{n-1} \) for all integers \( n \geq 0 \). You may assume and use the following facts: the product rule is true and the derivative of a constant is 0 (this can be shown to be true geometrically), as long as you state when you use them.

The Well-Ordering Principle

Theorem 3 (Well-Ordering Principle (WOP)). Every nonempty subset of \( \mathbb{N} \) has a least element.

Theorem 4. The WOP is equivalent to PMI and 2PMI

Example 8. Use the WOP to prove the Division Algorithm.