The Hat Check Problem

Where’s my hat?

A group of $n$ hat-wearing people enter a restaurant and leave their hats with the hat check girl, who is new on the job and forgets to put the ticket stubs with the correct hats. As a result, when the group leaves everyone is given a random hat from the $n$ hats. What is the probability that no one gets his own hat back?

If we view this through the lens of permutations, we are trying to find the probability that a random permutation (on a set $A = \{a_1, a_2, \ldots, a_n\}$ of $n$ elements) contains
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If we view this through the lens of permutations, we are trying to find the probability that a random permutation (on a set $A = \{a_1, a_2, \ldots, a_n\}$ of $n$ elements) contains no fixed points.
Let $A_i$ be the event “$a_i$ is a fixed point,” i.e., “person $a_i$ gets his own hat back.” Then
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$$P(A_i) = \frac{1 \cdot (n - 1) \cdot (n - 2) \cdot \cdots \cdot 1}{\text{total number of permutations}}$$

$$= \frac{1 \cdot (n - 1)!}{n!}$$

$$= \frac{1}{n}$$
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**Note:** This includes the permutations where people other than $a_i$ get their own hats back, too.
Similarly, the probability that [at least] people $a_i$ and $a_j$ are fixed points (i.e., at least $a_i$ and $a_j$ get their hats back) is
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\[
P(A_i \cap A_j) = \frac{1 \cdot 1 \cdot (n - 2) \cdot (n - 3) \cdots \cdot 1}{n!} \\
= \frac{(n - 2)!}{n!} \\
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So to find the probability of having at least one fixed point, we use inclusion/exclusion:

$$P(\text{at least one fp}) = \text{(ways to ensure one fp)} - \text{(ways to ensure two fp)}$$

$$+ \cdots + (-1)^{n-1}\text{(ways to ensure n fp)}$$
This yields

\[ P(\# fp \geq 1) = \left( \frac{n}{1} \right) \frac{1}{n} - \left( \frac{n}{2} \right) \frac{1}{n(n-1)} + \left( \frac{n}{3} \right) \frac{1}{n(n-1)(n-2)} - \]

\[ \cdots + (-1)^{n-1} \left( \frac{n}{n} \right) \frac{1}{n!} \]

\[ = n \frac{1}{n} - \frac{n!}{2!(n-2)!} \frac{1}{n(n-1)} + \frac{n!}{3!(n-3)!} \frac{1}{n(n-1)(n-2)} - \]

\[ \cdots + (-1)^{n-1} \frac{1}{n!} \]

\[ = 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{n-1} \frac{1}{n!} \]
This yields

\[ P(\#fp \geq 1) = \binom{n}{1} \frac{1}{n} - \binom{n}{2} \frac{1}{n(n-1)} + \binom{n}{3} \frac{1}{n(n-1)(n-2)} - \]

\[ \cdots + (-1)^{n-1} \binom{n}{n} \frac{1}{n!} \]

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So

\[ P(\#fp = 0) = 1 - P(\#fp \geq 1) \]

\[ = 1 - \left( 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \cdots + (-1)^{n-1} \frac{1}{n!} \right) \]
\[
\frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!}
\]
\[
\approx \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots + (-1)^n \frac{1}{n!} + \ldots
\]  
(Make an infinite series.)

\[= e^{-1} \]

\[\approx .368 \]

The finite sum and the infinite series will be very close to each other for \( n \geq 6 \), so no matter how many people we have over 6 the probability that no one gets their hat back is always about 36.8%. 
Is this deck random?

When shuffling a deck how many times must we shuffle in order for the deck to be “random?”

If we have no prior knowledge about the deck, then it essentially has a random order as far as we are concerned – if we were to choose a card at random from the deck, then we have a \( \frac{1}{52} \) probability of guessing what that card will be.

Similarly, if we wanted to draw a particular card from the deck then we would have a \( \frac{1}{52} \) probability of selecting our desired card from the deck.

However, if we had perfect knowledge of the state of the deck, then both of those probabilities would be 1.
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Random: An ordering of the deck is random if our knowledge about what card will be drawn (or where a card is located) is equivalent to the uniform distribution.
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Riffle Shuffle

Definition

We denote the probability distribution on a deck of $n$ cards after $k$ riffle shuffles by $\text{Rif}^k$. 

Theorem

\[
\|\text{Rif}^k - U\| \leq 1 - \frac{1}{n} \prod_{i=1}^{k} \left(1 - \frac{i}{2}\right)
\]
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[Skip a lot of work]
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\[
\| \text{Rif}^{*k} - U \| \leq 1 - \prod_{i=1}^{n-1} \left( 1 - \frac{i}{2^k} \right)
\]
For \( n = 52 \) cards and \( k = 1, \ldots, 20 \) shuffles we have the following:

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<tr>
<th>( k )</th>
<th>Variation Distance</th>
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<tr>
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<tr>
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<td>1.000000</td>
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<td>0.001263794</td>
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