THE STRONG SYMMETRIC GENUS OF THE FINITE COXETER GROUPS

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1. Introduction

It is known that the automorphism group of a compact Riemann surface of genus $g \geq 2$ is finite; furthermore, the size of this automorphism group must be no larger than $84(g-1)$ when $g \geq 2$ [11]. When a finite group G can be realized as an automorphism group of a genus g compact Riemann surface such that $|G| = 84(g-1)$, we say that G is a Hurwitz group. If a group G is not the automorphism group of any Riemann surface of genus 0 or 1, then for any Riemann surface, on which the group acts as an automorphism group the genus, must be at least $1+\frac{|G|}{84}$. Burnside [2] began the investigation of a related problem: find the the least genus g of a Riemann surface on which a given finite group acts faithfully as a group of automorphisms. Equivalently, given a finite group G, one can work find the least genus g of a closed orientable topological surface on which G acts as a group of orientation preserving symmetries. The latter parameter is denoted $\sigma^0(G)$ and has become known as the strong symmetric genus of G (see [10, Chapter 6]).

Many results are known concerning the strong symmetric genus of finite groups. All finite groups with strong symmetric genus less than 4 are known [1, 13]. It has been shown as well that for each positive integer n, there is a finite group G with $\sigma^0(G) = n$ [14] . The strong symmetric genus of several infinite families of finite groups have been found: the alternating and symmetric groups [3, 4, 5], the hyperoctahedral groups [12], the groups $PSL_2(q)$ [8, 9], and the groups $SL_2(q)$ [19]. In addition, the strong symmetric genus has been found for the sporadic finite simple groups [6, 20, 21, 22].

The symmetric groups and the hyperoctahedral groups are two infinite families of finite Coxeter groups. They are often referred to as the A-type and the B-type Coxeter groups, respectively. As stated above, the strong symmetric genus is known for each group in these families. Another family of finite Coxeter groups is the dihedral groups, which are automorphism groups of the sphere and, thus, have strong symmetric genus 0. This leaves one infinite family of finite Coxeter groups whose strong symmetric genus has not been found previously: the D-type groups. In this paper we calculate the strong symmetric genus of the D-type finite Coxeter groups and of the sporadic finite Coxeter groups. These new results will be given in Theorem 1 below; the previously known results concerning the other finite Coxeter groups will also be stated in Theorem 1.

The strong symmetric genus of the finite Coxeter groups will be shown by demonstrating the existence of certain pairs of generators. If a finite group G has generators x and y of orders p and q, respectively, with xy having order r, then we say that (x, y) is a (p, q, r) generating pair of G. By the obvious symmetries concerning generators, we will use the convention that $p \leq q \leq r$. Following the convention of Marston Conder [5], we say that a (p, q, r) generating pair of G is a minimal generating pair if there does not exist a (k, l, m) generating pair for G with $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$.

Before stating Theorem 1, we define the groups involved. For an $n \geq 3$, B_n will be the group of symmetries of the n dimensional cube, and D_n will be the group of orientation preserving symmetries of the n dimensional cube. For the sporadic finite Coxeter groups, we list the results in Table I.

Theorem 1. *Let G be a finite Coxeter group. If G is the dihedral group of size $2n$, then G has a $(2, 2, n)$ minimal generating pair and $\sigma^0(G) = 0$. If $G = \Sigma_n$ for $n > 29$, then G has a $(2, 3, 8)$ minimal generating pair and $\sigma^0(\Sigma_n) = \frac{n!}{48}$. If $G = B_n$ for $n > 8$, then G has a $(2, 4, 6)$ minimal generating pair and $\sigma^0(B_n) = \frac{n!\,2^n}{24}$. If $G = D_n$ for $n > 29$, then G has a $(2, 3, 8)$ minimal generating pair and $\sigma^0(D_n) = \frac{n!\,2^{n-1}}{48}$. The results for G being one of the sporadic cases are given in Table II. The remaining cases are listed in Table III.*

2. Generating Pairs and Strong Symmetric Genus

Recall that the groups of small strong symmetric genus are well known (see [1, 13]). The only finite Coxeter groups G with $\sigma^0(G) = 0$ are the dihedral groups, G_2, Σ_3, Σ_4, and D_3. Also there are no finite Coxeter groups G with $\sigma^0(G) = 1$. In this paper, we will assume that $\sigma^0(G) > 1$ for each group G that we are discussing. It is known that for groups with $\sigma^0(G) > 1$, any generating pair will be a (p, q, r) generating pair with $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$. Using the Riemann-Hurwitz equation, we see that given any generating pair of G, we get an upper bound on the strong symmetric genus of G [17]. If G has a (p, q, r) generating pair, then $\sigma^0(G) \leq 1 + \frac{1}{2}|G| \cdot (1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r})$. The following lemma, which is a result of Singerman [16] (see also [13, 18]), shows that the strong symmetric genus for many groups is computed directly from a minimal generating pair.

Lemma 2 (Singerman [16]). *Let G be a finite group such that $\sigma^0(G) > 1$. If $|G| > 12(\sigma^0(G) - 1)$, then G has a (p, q, r) generating pair with*

$$\sigma^0(G) = 1 + \frac{1}{2}|G| \cdot (1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}). \tag{1}$$

In addition, if $p \neq 2$, $p = q = 3$, and r is 4 or 5.

We also include a lemma that allows for control of minimal generating pairs when passing to quotient groups.

Lemma 3. *Let G be a finite group such that G has a (p, q, r) generating pair. For any normal subgroup $N \lhd G$, any minimal generating pair of G/N must have a (p', q', r') generating pair such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \leq \frac{1}{p'}+\frac{1}{q'}+\frac{1}{r'}$. In other words, $\sigma^0(G/N)-1 \leq \frac{\sigma^0(G)-1}{|N|}$.*

Proof. Suppose that (x, y) is a (p, q, r) generating pair of G. In addition, let $\bar{x}$ and $\bar{y}$ be the images under the quotient map $\pi : G \to G/N$ of x and y respectively. Let p', q', and r' be the orders of $\bar{x}$, $\bar{y}$, and $\bar{x}\bar{y}$. Clearly, $p' \leq p$, $q' \leq q$, and $r' \leq r$; therefore, $\frac{1}{p'} + \frac{1}{q'} + \frac{1}{r'} \geq \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$. It is also clear that $\bar{x}$ and $\bar{y}$ generate G/N. □

In order to use this quotient group result, we recall some cases where certain Coxeter groups are quotients of other Coxeter groups: For a fixed n, $\Sigma_n \cong B_n/(\mathbb{Z}_n)^n$, and $\Sigma_n \cong D_n/(\mathbb{Z}_n)^{n-1}$. In addition if n is odd, $D_n \cong B_n/Z(B_n)$.

For an example of using quotient groups, we look at D_{17}. We notice that both B_{17} and Σ_{17} have minimal $(2, 4, 6)$ generating pairs. D_{17} has a minimal generating pair; suppose D_{17} has a (p, q, r) minimal generating pair. Since $D_{17} \cong B_{17}/Z(B_{17})$, $\frac{11}{12} \leq \frac{1}{p}+\frac{1}{q}+\frac{1}{r}$ with $p \leq 2$, $q \leq 4$, and $r \leq 6$. On the other hand, we have $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \leq \frac{11}{12}$ because $\Sigma_{17} \cong D_{17}/(\mathbb{Z}_2)^{16}$. We see that D_{17} must have a $(2, 4, 6)$ minimal generating pair.

3. Generators of D_n

For notation purposes, we recall that D_n is an index two subgroup of B_n and that B_n is the wreath product $\mathbb{Z}_2 \wr \Sigma_n$. We will use notation that was adopted by V. S. Sikora [15] for the elements of B_n and thus D_n (see also [12]). For an element of B_n, we will write a tuple $[\sigma, b]$ where σ is an element of Σ_n and where b is a list of n binary digits representing the element of $(\mathbb{Z}_2)^n$. The multiplication then becomes $[\sigma, b] \cdot [\tau, c] = [\sigma \cdot \tau, \tau^{-1}(b) + c]$ where addition in the binary digits is a parity computation in each entry. We will use the convention of calling b even or odd according to the number of ones appearing as binary digits of b. Notice that if b and c have the same parity, then $b + c$ is even; and if they differ in parity, then $b + c$ is odd. Using this notation, an element $[\sigma, b] \in B_n$ is contained in D_n if and only if b is even.

Notice that if $[\sigma, b]$ and $[\tau, c]$ generate D_n, then σ and τ must generate Σ_n. Since we are looking to find generators of D_n, we need to first find generators of Σ_n. Next we construct generators of D_n from the generators of Σ_n. Also during this construction we would like to have control of the orders of the generators of D_n as well as to have control of the order of their product as described in Section 2.

Recall that the following sequence is a split exact sequence of groups:

$$(\mathbb{Z}_2)^{n-1} \xrightarrow{i} D_n \xrightarrow{\pi} \Sigma_n \tag{2}$$

where $i(b) = [1, b]$ and $\pi([\sigma, b]) = \sigma$. In addition, we have the following commutative diagram

$$\begin{array}{ccccc} (\mathbb{Z}_2)^{n-1} & \xrightarrow{i} & D_n & \xrightarrow{\pi} & \Sigma_n \\ \downarrow & & \downarrow & & \| \\ (\mathbb{Z}_2)^{n} & \xrightarrow{i} & B_n & \xrightarrow{\pi} & \Sigma_n \end{array} \tag{3}$$

where both horizontal sequences are split exact. Using the bottom split exact sequence, the author has proven the following proposition:

Proposition 4 (Jackson [12]). *For $n \geq 5$, let G be a subgroup of B_n with $\pi(G) = \Sigma_n$; then G is a split extension of Σ_n by one of the following: 1, $\mathbb{Z}_2$, $(\mathbb{Z}_2)^{n-1}$, or $(\mathbb{Z}_2)^n$. In the first two cases, G is isomorphic to Σ_n and $\mathbb{Z}_2 \times \Sigma_n$, respectively; in the third case either*

$$G = \{[\sigma, b] \in B_n | b \text{ is even}\} = D_n \text{ or } G = \left\{ [\sigma, b] \in B_n \,\middle|\, \begin{array}{c} b \text{ is even if } \sigma \in A_n \\ b \text{ is odd if } \sigma \in \Sigma_n \setminus A_n \end{array} \right\};$$

in the fourth case $G = B_n$.

Proposition 4 leads to Corollary 5:

Corollary 5. *For $n \geq 5$, let H be a subgroup of D_n with $\pi(H) = \Sigma_n$; then H is a split extension of Σ_n by one of the following: 1, $\mathbb{Z}_2$, or $(\mathbb{Z}_2)^{n-1}$. In the first case, $H \cong \Sigma_n$. In the second case, which can only occur when n is even, $H \cong \Sigma_n \times Z(D_n)$. In the third case, $H = D_n$.*

Using Corollary 5, we can prove Proposition 6, which we will use to construct generators of D_n from those of Σ_n.

Proposition 6. *Suppose σ and τ generate Σ_n as $Symm(\Gamma)$ where $\Gamma = \{1, 2, \ldots, n\}$ with both σ and $\sigma \cdot \tau$ having even order. Assume, furthermore, that σ fixes two elements i and j of Γ, which are in the same cycle of the element $\sigma \cdot \tau$; assume as well that σ fixes a third element of Γ if n is even. Let $b = (0, 0, \ldots, 0, 0)$, and let $a = (0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0)$ where there is a 1 in the i^{th} position and in* *the j^{th} position. Under these conditions, $[\sigma, a]$ and $[\tau, b]$ generate D_n. In addition the elements $[\sigma, a]$, $[\tau, b]$ and $[\sigma \cdot \tau, \tau^{-1}(a)]$ have the same orders as σ, τ, and $\sigma \cdot \tau$, respectively.*

Proof. The result concerning the orders of $[\sigma, a]$ and $[\tau, b]$ are obvious since $\sigma(a) = a$ and $\tau(b) = b$. On the other hand, $(\sigma \cdot \tau)^{-k}(\tau^{-1}(a)) = (\sigma \cdot \tau)^{-k-1}(a)$. Notice that if k is the length of the cycle in $\sigma \cdot \tau$ that contains i and j, then

$$[\sigma \cdot \tau, \tau^{-1}(a)]^k = [(\sigma \cdot \tau)^k, (0, 0, \ldots, 0, 0)]. \tag{4}$$

It is clear then that the order of $[\sigma \cdot \tau, \tau^{-1}(a)]$ is the same as the order of $\sigma \cdot \tau$.

Let $H = \langle [\sigma, a], [\tau, b] \rangle \subset D_n$. We need to show that $H = D_n$. Notice that H is a subgroup of D_n such that $\pi(H) = \Sigma_n$. From Corollary 5 we know that H is a split extension of Σ_n by one of the following: 1, $\mathbb{Z}_2$, or $(\mathbb{Z}_2)^{n-1}$; we know that the second case only occurs when n is even.

Recall that any section $s : \Sigma_n \to D_n$ takes $\alpha \in A_n$ to $[1, d] \cdot [\alpha, (0, \ldots, 0)] \cdot [1, d]^{-1}$ for some $[1, d] \in B_n$; and if $\alpha \in \Sigma_n \setminus A_n$, then either $s(\alpha) = [1, d] \cdot [\alpha, (0, \ldots, 0)] \cdot [1, d]^{-1}$ or $s(\alpha) = [1, d] \cdot [\alpha, (1, \ldots, 1)] \cdot [1, d]^{-1}$. Also notice that the last case only occurs if n is even.

We will show first that $[\sigma, a]$ cannot be in the image of any such section via contradiction. Suppose $[\sigma, a]$ is in the image of some section homorphism $s : \Sigma_n \to D_n$. Now $[\sigma, a]$ cannot be $[1, d] \cdot [\sigma, (0, \ldots, 0)] \cdot [1, d]^{-1} = [\sigma, \sigma^{-1}(d) + d]$ for any $[1, d] \in B_n$ since σ fixes i and j so that $\sigma^{-1}(d) + d$ has a 0 in both the i^{th} and j^{th} positions. We, therefore, may assume that n is even and

$$[\sigma, a] = [1, d] \cdot [\sigma, (1, \ldots, 1)] \cdot [1, d]^{-1} = [\sigma, \sigma^{-1}(d) + d + (1, \ldots, 1)] \tag{5}$$

for some $[1, d] \in B_n$. $\sigma^{-1}(d) + d + (1, \ldots, 1) \neq a$ since σ fixes k and a has a zero in the k^{th} position.

$[\sigma, a]$ is not in the image of any section; thus we only need to show that $[\sigma, a]$ is not equal to $s(\sigma) \cdot [1, (1, \ldots, 1)]$ for any section s. Notice that if $[\sigma, a]$ were such an element, then a is either $\sigma^{-1}(d) + d + (1, \ldots, 1)$ or $\sigma^{-1}(d) + d$, which are both ruled out in the previous paragraph. □

4. Results

From Section 3, we see that the strong symmetric genus of D_n may be shown by demonstrating a particular generating pair of Σ_n. First we notice that if Σ_n has a (p, q, r) generating pair, then at most one of p, q, or r is odd. This result also holds for generators of D_n. So we see that the best possible generating pair for any D_n with $n > 3$ is a $(2, 3, 8)$ generating pair. We notice that Marston Conder [3] has demonstrated $(2, 3, 8)$ generating pairs for each Σ_n with $n \geq 168$. If we call this generating pair (σ, τ), we notice that for each $n \geq 168$, excluding those for values of n listed below, σ fixes three elements, two of which are in the same cycle of $\sigma \cdot \tau$. In these cases we apply Proposition 6 to see that D_n has a $(2, 3, 8)$ minimal generating pair. The exceptional values of n are 171, 173, 174, 181, 185, 188, 194, 201, 202, 206, 209, 214, 230, 250, 257, 265, and 286.

In the remaining cases, the results were computed using GAP [7]. For $n \geq 30$, including the exceptional values of n listed above, D_n was shown to have a $(2, 3, 8)$ minimal generating pair. So for each $n \geq 30$, $\sigma^0(D_n) = \frac{n!\, 2^{n-1}}{48}$. For each D_n with $n < 30$ as well as for each sporadic group listed in Table I, an exhustive search was performed using GAP [7] to find a minimal generating pair. In this manner the new results found in Table I and Table II, as well as in Theorem 1, were obtained.

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group G	description
G_2	isomorphic to the dihedral group of order 12
I_3	symmetry group of the regular icosahedron
I_4	symmetry group of the 4- dimensional, regular 120-cell
F_4	symmetry group of the 4-dimensional, regular 24-cell
E_6	symmetry group of the E_6 polytope
E_7	symmetry group of the E_7 polytope
E_8	symmetry group of the E_8 polytope

Table I. Sporadic Coxeter group descriptions

group G	size	min. gen. pair	$\sigma^0(G)$
G_2	12	$(2, 2, 6)$	0
I_3	120	$(2, 3, 10)$	5
I_4	14400	$(2, 4, 6)$	601
F_4	1152	$(2, 6, 6)$	97
E_6	$51, 840$	$(2, 4, 8)$	3241
E_7	$2, 903, 040$	$(2, 4, 7)$	$155, 521$
E_8	$697, 729, 600$	$(2, 4, 8)$	$43, 545, 601$

TABLE II. Sporadic Coxeter group results

n	Σ_n best Δ	Σ_n $\sigma^0(\Sigma_n)$	B_n best Δ	B_n $\sigma^0(B_n)$	D_n best Δ	D_n $\sigma^0(D_n)$
$n=3$	Dihedral of order 6		$(2,4,6)$	$\frac{3!2^3}{24}+1=3$	$D_3=\Sigma_4$	
$n=4$	$(2,3,4)$	0	$(2,4,6)$	$\frac{4!2^4}{24}+1=17$	$(3,4,4)$	$\frac{4!2^3}{12}+1=17$
$n=5$	$(2,4,5)$	$\frac{5!}{40}+1=4$	$(2,4,10)$	$\frac{5!2^5(3)}{40}+1$	$(2,4,5)$	$\frac{5!2^4}{40}+1$
$n=6$	$(2,5,6)$	$\frac{6!}{15}+1$	$(2,6,6)$	$\frac{6!2^6}{12}+1$	$(2,5,6)$	$\frac{6!2^5}{15}+1$
$n=7$	$(2,3,10)$	$\frac{7!}{30}+1$	$(2,4,6)$	$\frac{7!2^7}{24}+1$	$(2,4,6)$	$\frac{7!2^6}{24}+1$
$n=8$	$(2,4,7)$	$\frac{8!(3)}{56}+1$	$(2,4,8)$	$\frac{8!2^8}{16}+1$	$(2,4,7)$	$\frac{8!2^7(3)}{56}+1$
$n=9$	$(2,4,6)$	$\frac{9!}{24}+1$	$(2,4,6)$	$\frac{9!2^9}{24}+1$	$(2,4,6)$	$\frac{9!2^8}{24}+1$
$n=10$	$(2,3,10)$	$\frac{10!}{30}+1$	$(2,4,6)$	$\frac{10!2^{10}}{24}+1$	$(2,3,10)$	$\frac{10!2^9}{30}+1$
$n=11$	$(2,4,5)$	$\frac{11!}{40}+1$	$(2,4,6)$	$\frac{11!2^{11}}{24}+1$	$(2,4,5)$	$\frac{11!2^{10}}{40}+1$
$n=12$	$(2,3,12)$	$\frac{12!}{24}+1$	$(2,4,6)$	$\frac{12!2^{12}}{24}+1$	$(2,3,12)$	$\frac{12!2^{11}}{24}+1$
$n=13$	$(2,3,12)$	$\frac{13!}{24}+1$	$(2,4,6)$	$\frac{13!2^{13}}{24}+1$	$(2,3,12)$	$\frac{13!2^{12}}{24}+1$
$n=14$	$(2,4,6)$	$\frac{14!}{24}+1$	$(2,4,6)$	$\frac{14!2^{14}}{24}+1$	$(2,3,14)$	$\frac{14!2^{13}}{21}+1$
$n=15$	$(2,4,5)$	$\frac{15!}{40}+1$	$(2,4,6)$	$\frac{15!2^{15}}{24}+1$	$(2,4,5)$	$\frac{15!2^{14}}{40}+1$
$n=16$	$(2,4,5)$	$\frac{16!}{40}+1$	$(2,4,6)$	$\frac{16!2^{16}}{24}+1$	$(2,4,5)$	$\frac{16!2^{15}}{40}+1$
$n=17$	$(2,4,6)$	$\frac{17!}{24}+1$	$(2,4,6)$	$\frac{17!2^{17}}{24}+1$	$(2,4,6)$	$\frac{17!2^{16}}{24}+1$
$n=20$	$(2,3,8)$	$\frac{20!}{48}+1$	$(2,4,6)$	$\frac{20!2^{20}}{24}+1$	$(2,4,5)$	$\frac{20!2^{19}}{40}+1$
$n=22$	$(2,3,10)$	$\frac{22!}{30}+1$	$(2,4,6)$	$\frac{22!2^{22}}{24}+1$	$(2,3,10)$	$\frac{22!2^{21}}{30}+1$
$n=23$	$(2,3,10)$	$\frac{23!}{30}+1$	$(2,4,6)$	$\frac{23!2^{23}}{24}+1$	$(2,3,12)$	$\frac{23!2^{22}}{24}+1$
$n=26$	$(2,4,5)$	$\frac{26!}{40}+1$	$(2,4,6)$	$\frac{26!2^{26}}{24}+1$	$(2,4,5)$	$\frac{26!2^{25}}{40}+1$
$n=29$	$(2,3,12)$	$\frac{29!}{24}+1$	$(2,4,6)$	$\frac{29!2^{29}}{24}+1$	$(2,3,12)$	$\frac{29!2^{28}}{24}+1$

TABLE III. Exceptional case results