Recall that the $p$-rank of a finite group $G$, $\text{rk}_p(G)$, is the largest rank of an elementary abelian $p$-subgroup of $G$ and that the rank of a finite group $G$, $\text{rk}(G)$, is the maximum of $\text{rk}_p(G)$ taken over all primes $p$. We define the homotopy rank of a finite group $G$, $h(G)$, to be the minimal integer $k$ such that $G$ acts freely on a finite CW-complex $Y \cong S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$. Benson and Carlson [3] have conjectured that for any finite group $G$, $\text{rk}(G) = h(G)$. The case of their conjecture, when $G$ is a rank one group, is a direct result of Swan’s theorem [9]. Benson and Carlson’s conjecture has also been verified by Adem and Smith [1] for rank two $p$-groups as well as for all rank two finite simple groups except $\text{PSL}_3(\mathbb{F}_p)$ for $p$ an odd prime.

Recall that $\text{Qd}(\mathbb{F}_p)$ is the semidirect product of $(\mathbb{Z}/p\mathbb{Z})^2$ by $\text{SL}_2(\mathbb{F}_p)$ with the natural action. We say that a group $G$ does not involve $\text{Qd}(\mathbb{F}_p)$ if no subquotient of $G$ is isomorphic to $\text{Qd}(\mathbb{F}_p)$. In addition, a group that does not involve $\text{Qd}(\mathbb{F}_p)$ is called $\text{Qd}(\mathbb{F}_p)$-free. Here we will verify Benson and Carlson’s conjecture for $\text{Qd}(\mathbb{F}_p)$-free finite groups of rank two. A result of Heller [5] states that if $h(G) = 2$, then $\text{rk}(G) = 2$; therefore, the conjecture holds for a given rank two group $G$, if $G$ acts freely on a finite CW-complex $Y \cong S^{n_1} \times S^{n_2}$. We will find such actions using a recent result of Adem and Smith, but we need to include two definitions before stating their result.

**Definition 1.** Let $\varphi : BG \to BU(n)$ be a map and let $\alpha \in H^{2n}(BU(n), \mathbb{Z})$ be the Euler class (top Chern class) of $U(n)$. The Euler class in $H^{2n}(BG, \mathbb{Z})$ associated to $\varphi$ is $\varphi^*(\alpha)$.

**Definition 2.** A cohomology class $\alpha \in H^*(BG, \mathbb{Z})$ is called effective if for each elementary abelian subgroup $E \subseteq G$ with $\text{rk}(E) = \text{rk}(G)$, $\text{res}^G_E(\alpha) \neq 0$.

**Theorem 3** (Adem and Smith [1]). Let $G$ be a finite group with $\text{rk}(G) = 2$. If the Euler class associated to some map $\varphi : BG \to BU(n)$ is effective, then $h(G) = 2$.

In light of Theorem 3, verifying Benson and Carlson’s conjecture for a rank two group $G$ can be reduced to finding a particular map $\varphi : BG \to BU(n)$ with an effective Euler class. Two properties of maps from $BG$ to $BU(n)$ will prove useful: homotopic maps have the same Euler class; and if $G$ is a $p$-group for some prime $p$, then maps from $BG$ to $BU(n)$ are in one-to-one correspondence with complex characters of degree $n$. The second property uses a result of Dwyer and Zabrodsky [4].

In light of the second property above, we will be relating maps from $BG$ to $BU(n)$ to characters. To do so, we must introduce the following notation: $G_p$ will denote a Sylow $p$-subgroup of $G$; $\text{Char}_n(G_p)$ will denote the set of degree $n$ complex characters of $G_p$; and $\text{Char}_n^G(G_p)$ will denote the subset of $\text{Char}_n(G_p)$ consisting of those degree $n$ complex characters of $G_p$ that are the restrictions of class functions on $G$, meaning that they respect fusion in $G$.
We now define a map $\psi_G : [BG, BU(n)] \to \prod_{p||G} \text{Char}^G_n(G_p)$ by using the following composition:

$$[BG, BU(n)] \xrightarrow{\cong} \prod_{p||G} [BG, BU(n)]_p \xrightarrow{\cong} \prod_{p||G} [BG_p, BU(n)]_p \xrightarrow{\cong} \prod_{p||G} \text{Char}_n(G_p).$$

Notice that spaces in the center and at the right of the top row contain $BU(n)_p$, which is the $p$-completion of the space $BU(n)$. The left bijection is a result by Jackowski, McClure, and Oliver [6] while the right bijection follows is the previously mentioned property. The restriction map $\text{res}$ is induced by the inclusion of the Sylow $p$-subgroups $G_p$ into $G$. We notice that the image of the composition above lies in the subset $\prod_{p||G} \text{Char}^G_n(G_p)$; therefore, we let $\psi_G$ be the composition above with the range restricted to $\prod_{p||G} \text{Char}_n(G_p)$. We get the following result concerning the map $\psi_G$:

**Theorem 4** (Jackson [7, Theorem 1.3]). If $G$ is a finite group of rank two, then the natural mapping $\psi_G : [BG, BU(n)] \to \prod_{p||G} \text{Char}^G_n(G_p)$ is a surjection.

Using Theorem 4, we see that a map from $BG$ to $BU(n)$ with an effective Euler class can be demonstrated by giving appropriate characters in $\text{Char}^G_n(G_p)$ for each prime $p$ dividing the order of $G$, which leads to Definition 5 and Theorem 6

**Definition 5.** Let $G$ be a finite group, $p$ a prime dividing $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. A character $\chi$ of $G_p$ is called a $p$-effective character of $G$ if $\chi \in \text{Char}^G_n(G_p)$ and if for each elementary abelian subgroup $E \subseteq G_p$ with $\text{rk}(E) = \text{rk}(G)$, the trivial character of $E$ is not an irreducible summand of the character $\chi|E$.

**Theorem 6** (Jackson [8]). Let $G$ be a finite group. If for each prime $p$ dividing $|G|$ there exists a $p$-effective character of $G$, then there is a map $\varphi : BG \to BU(n)$ whose associated Euler class is effective. If in addition $\text{rk}(G) = 2$, then $b(G) = 2$.

Theorem 6 has reduced the process of showing that a rank two group has homotopy rank two to finding $p$-effective characters for each prime $p$ dividing the order of the group. A definition from group theory is necessary in demonstrating the existence of $p$-effective characters.

**Definition 7.** Let $G$ be a finite group, and let $H$ and $K$ be subgroups such that $H \subseteq K$. We say that $H$ is strongly closed in $K$ with respect to $G$ if for each $g \in G$, $H^g \cap K \subseteq H$.

We are now able to show a sufficient condition for the existence of a $p$-effective character.

**Proposition 8.** Let $G$ be a finite group, $p$ a prime divisor of $|G|$, and $G_p$ a Sylow $p$-subgroup of $G$. If $H \subseteq \mathbb{Z}(G_p)$ exists such that $H$ is non-trivial and strongly closed in $G_p$ with respect to $G$, then $G$ has a $p$-effective character.

Recall that $\Omega_1(P)$, for a $p$-group $P$, is the subgroup of $P$ generated by the order $p$ elements of $P$. Notice that if $P$ is abelian, then $\Omega_1(P)$ is elementary abelian. The next theorem shows that in many cases the sufficient condition may be applied.
Theorem 9 (Jackson [8]). Let $G$ be a finite group, $p$ a prime with $\text{rk}_p(G) = \text{rk}(G) = 2$, and $G_p \in \text{Syl}_p(G)$. If $\Omega_1(Z(G_p))$ is not strongly closed in $G_p$ with respect to $G$, then either $p$ is odd and $Qd(p)$ is involved in $G$, or $p = 2$ and $G_2$ is dihedral, semi-dihedral, or wreathed.

The prime 2 portion of Theorem 9 is a result of Alperin, Brauer, and Gorenstein [2, Proposition 7.1]. As a result of Theorem 9, a rank two finite group has a 2-effective character if its Sylow 2-subgroups are not dihedral, semi-dihedral or wreathed. The cases of dihedral, semi-dihedral and wreathed Sylow 2-subgroups are shown to have 2-effective characters in Theorem 9.

Theorem 10 (Jackson [8]). If $G$ is a finite group with a dihedral, semi-dihedral, or wreathed Sylow 2-subgroup such that $\text{rk}(G) = 2$, then $G$ has a 2-effective character.

Theorem 10 is shown by explicitly demonstrating the 2-effective character in each case.

Theorem 11 (Jackson). Let $G$ be a finite group such that $\text{rk}(G) = 2$. $G$ acts freely on a finite CW-complex $Y \simeq S^{n_1} \times S^{n_2}$ unless for some odd prime $p$, $G$ involves $Qd(p)$. In particular, if $G$ is a rank two group that is $Qd(p)$-free for each odd prime $p$, then $h(G) = 2$.

We end by pointing out that for an odd prime $p$, $Qd(p)$ does not have a $p$-effective character.

References