THE STRONG SYMMETRIC GENUS OF SMALL GENERALIZED SYMMETRIC GROUPS

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The Generalized Symmetric Groups

Permutation Matrices

A permutation matrix is an square matrix where each row and column has exactly one "1" and all other entries are "0's". The group of all $n \times n$ permutation matrices is isomorphic to the symmetric group Σ_n.

An element from Σ_n correspond to a particular permutation matrix based on the permutation of the columns from the identity matrix.

Example: Consider $(1\ 2\ 3) \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Roots of Unity

An m^{th}-root of unity is a complex number z that satisfies the equation $z^m = 1$.

The m^{th}-roots of unity are $e^{2k\pi i/m}$ for $k \in \{0, 1, 2, \ldots, m-1\}$

We see that there are m m^{th} roots of unity.

Let $\alpha = e^{2\pi i/m}$. We can now consider the m^{th} roots of unity to be α^k for $k \in \{0, 1, \ldots, m-1\}$.

Notice that $C_m =< \alpha >= \{\alpha, \alpha^2, \alpha^3, \ldots, \alpha^m = \alpha^0 = 1\}$ is a cylic group of size m.

The Generalized Symmetric Group

$$G(n,m) = C_m \wr \Sigma_n = (C_m)^n \rtimes \Sigma_n = \underbrace{C_m \times C_m \times \cdots \times C_m}_{n \text{ times}} \rtimes \Sigma_n$$

$G(n,m) \simeq HK \leq GL_n(\mathbb{C})$ where
$H = \{$group of all $n \times n$ diagonal matrices with entries in $C_m\}$
$K = \{$group of all $n \times n$ permutation matrices$\}$

So $G(n,m)$ for $n, m \in Z^+$ is the group of all $n \times n$ pseudo-permutation matrices with m^{th} roots of unity as the nonzero entries. The group operation is matrix multiplication.

Notation

We describe a typical element of $G(n,m)$:
[(Symmetric Group Portion),(Cyclic Group Portion)]

$$\left[(1\ 3\ 2), (\alpha, 1, \alpha^2)\right] \rightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha^2 \\ \alpha & 0 & 0 \end{bmatrix} \in G(3,m)$$

$$[e, (\alpha, \alpha, \alpha, \alpha)] \rightarrow \begin{bmatrix} \alpha & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \alpha \end{bmatrix} \in G(4,m)$$

$$[(1\ 3), (1,1,1)] \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \in G(3,m)$$

Where the symmetric group portion is the cycle notation for an element of Σ_n and the cyclic group portion is a list of n elements in C_m. Notice that the elements in C_m correspond to the non-zero elements in the columns of the matrix.

Basic Outline of Project

We found the strong symmetric genus for the generalized symmetric groups $G(n,m)$ with n either 3, 4, or 5. The patterns were found using the computer algebra system Magma [1], but all results were proven without the use of a computer.

The Strong Symmetric Genus

The strong symmetric genus of a finite group G is the smallest genus of a compact orientable topological surface on which G acts faithfully as a group of orientation preserving symmetries.[5]

Riemann-Hurwitz Equation

Suppose there exists two elements of G, x and y that generates G. If x has order p, y has order q, and xy has order r, then there exists some surface on which G acts faithfully as a group of orientation preserving symmetries whose genus is as follows:

$$g = 1 + \frac{|G|}{2}\left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right).$$

Thus, in order to find the **smallest** genus, we wish to find the generating set $\{x, y\}$ that gives us the **largest** $\frac{1}{p}+\frac{1}{q}+\frac{1}{r}$. (This sum will turn out to always be < 1.) We refer to a generating set $\{x, y\}$ using its corresponding triple (p, q, r).

Results of Singerman and Jackson

For the generalized symmetric group $G = G(n,m)$ with $n > 3$ and $m \geq 3$, if $G(n,m)$ has an (p,q,r) generating pair with

$$(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}) \leq 1/3 \text{ when } m = 3 \text{ or } m = 6 \text{ and } (1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}) \leq 1/2 \text{ otherwise,}$$

then the strong symmetric genus of $G(n,m)$ is given by the Riemann-Hurwitz equation for some generating pair. [3,4,5]

Results

Triples giving the Strong Symmetric Genus

$\boldsymbol{G(3,m):}$ $(2, 2m, 3m)$ for all m

$\boldsymbol{G(4,m):}$ $(2, 2m, 3m)$ when $m \equiv 2 \pmod 4$
$(2, 3m, 4m)$ when $m \not\equiv 2 \pmod 4$

$\boldsymbol{G(5,m):}$ $(2, m, 4m)$ when $3 \mid m$ and $m \equiv 2 \pmod 4$
$(2, 2m, 4m)$ when $3 \mid m$ and $2 \not| \, m$
$(2, 2m, 5m)$ when $3 \not| \, m$ and $m \equiv 2 \pmod 4$
$(2, 4m, 5m)$ when $4 \mid m$ or both $3 \not| \, m$ and $2 \not| \, m$

General generators for $G(n,m)$ for any n and m

A = $[(1\ 2), (1, 1, \ldots, 1)]$, so $|\mathsf{A}| = 2$
B = $[(1\ 3\ 4\ \ldots\ n), (1, 1, 1, \ldots, 1, \alpha)]$, so $|\mathsf{B}| = (n-1)m$
BA = $[(1\ 2\ 3\ \ldots\ n), (1, 1, 1, \ldots, 1, \alpha)]$, so $|\mathsf{BA}| = nm$

The above generators provide the (p,q,r) triple: $(2, (n-1)m, nm)$ which provides the following upper bound for the strong symmetric genus of $G(n,m)$.

$$g = 1 + \frac{|G|}{2}\left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right) = 1 + \frac{m^n n!}{2}\left(1 - \frac{1}{2} - \frac{1}{(n-1)m} - \frac{1}{nm}\right) =$$

$$1 + \frac{m^n n!((n-1)(nm-2) - 2n)}{4(n-1)nm} = 1 + \frac{1}{4}(m^{n-1}(n-2)!((n-1)(nm-2) - 2n))$$

Approach to Solving the Problem

The basic approach to solving the problem: After demonstrating the generators with the smallest genus, show that any two elements of $G(n,m)$ which would produce a smaller genus cannot generate $G(n,m)$

Method 1 – The Elements Fail to Generate Σ_n

If the symmetric group parts of two elements reside in the same proper subgroup of Σ_n,then the symmetric group parts of these two elements cannot generate Σ_n; in particular the two elements cannot generate $G(n,m)$.

Sample Proper Subgroups:

I A_n – The set of even permutations in Σ_n is a proper normal subgroup of Σ_n
II D_n: $\langle a, b | a^2 = b^n = 1, aba = b^{-1} \rangle$
D_n is isomorphic to a proper subgroup of Σ_n
III Frobenius Group of order 20: $\langle s, t | s^4 = t^5 = 1, s^{-1}ts = t^2 \rangle$
The Frobenius Group of order 20 is isomorphic to a proper subgroup of Σ_5

Method 2 – The Elements Fail to Generate C_m

There is a homomorphism from $G(n,m) \rightarrow C_p$ where $p \mid m$
This homomorphism can be viewed as the two part process $G(n,m) \rightarrow C_m \rightarrow C_p$

Let $[\sigma, (\alpha^{a_1}, \alpha^{a_2}, \cdots, \alpha^{a_n})] \in G(n,m)$
$G(n,m) \rightarrow C_m$: $\alpha^{a_1+a_2+\ldots+a_n}$
$C_m \rightarrow C_p$: $\alpha^{\frac{m}{p}[a_1+a_2+\ldots+a_n (\text{mod } p)]}$

If both elements are in the Kernel of the mapping into C_2 or C_3 then they are both elements in a proper subgroup of $G(n,m)$

Method 3 – Conjugation into a Maximal Subgroup

Conjugation of two elements into the same proper maximal subgroup of $G(n,m)$, by a common conjugate, implies that these two elements are both in a conjugate maximal subgroup and thus do not generate $G(n,m)$

We conjugate into a proper maximal subgroup of the form:

$$\left\langle C_m \rtimes \Sigma_n, \left(C_{\frac{m}{p}}\right)^n \rtimes \Sigma_n \right\rangle.$$

Method 4 – Determinant

if m is even, then we will use the following mapping: $G(n,m) \xrightarrow{det} C_m \rightarrow C_2$.

If both elements are in the Kernel of this mapping, we know that they are both in a proper subgroup of $G(n,m)$ and thus do not generate $G(n,m)$.

References

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[5] D. Singerman. *Symmetries of Riemann surfaces with large automorphism group.* Math. Ann. **210** (1974), 17-32.