Most rank two finite groups act freely on a homotopy product of two spheres

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Conference on Pure and Applied Topology
Isle of Skye, Scotland
June 22, 2005

Notation and Statement of Problem:

Let G be a finite group.

$\mathrm{rk}_p(G)$ (p-rank) will be the largest rank of an elementary abelian p-subgroup of G.

$\mathrm{rk}(G)$ (rank) will be the maximum of $\mathrm{rk}_p(G)$ over all primes p.

Define $\mathrm{h}(G)$ (homotopy rank) to be the minimal integer k such that G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_k}$.

Benson and Carlson have conjectured that for any finite group G, $\mathrm{rk}(G) = \mathrm{h}(G)$.

When $\mathrm{rk}(G) = 1$, Swan's Thoerem verifies the conjecture.

Adem and Smith have also verified this conjecture for rank two p-groups and all rank two finite simple groups except $PSL_3(\mathbb{F}_p)$ for p an odd prime.

We will be focused on rank two groups today and will verify the conjecture for most rank two groups.

A result of A. Heller states that if h$(G) \leq 2$, then rk$(G) \leq 2$.

To verify the conjecture for rank two groups, we are left to show that each rank two group has homotopy rank two.

We will be able to do this for most rank two groups. Keep in mind the excluded case of Adem and Smith: $PSL_3(\mathbb{F}_p)$ for p an odd prime.

Results of Adem and Smith are heavily used in the present work through the following two Theorems:

Theorem 1 (Adem and Smith) *Let G be a finite group and let X be a finitely dominated, simply connected G-CW complex such that every nontrivial isotropy subgroup has rank one. Then for some large integer $N > 0$ there exists finite CW-complex $Y \simeq \mathbb{S}^N \times X$ and a free action of G on Y such that the projection $Y \to X$ is G-equivariant.*

Theorem 1 allows one to demonstrate that a group has homotopy rank two by showing that it acts on a finite CW-complex $X \simeq \mathbb{S}^m$ such that the nontrivial isotropy subgroups are rank one.

Definition 2 *Let* $\varphi : BG \to BU(n)$ *and let* $\alpha \in H^{2n}(BU(n), \mathbb{Z})$ *be the top Chern class (Euler class) of* $U(n)$*. The Euler class in* $H^{2n}(BG, \mathbb{Z})$ *associated to* φ *is* $\varphi^*(\alpha)$*.*

Definition 3 *A cohomology class* $\alpha \in H^*(BG, \mathbb{Z})$ *is called effective if for each* $E \subseteq G$ *an elementary abelian subgroup with* $\mathrm{rk}(E) = \mathrm{rk}(G)$, $\mathrm{res}^G_E(\alpha) \neq 0$*.*

Theorem 4 (Adem and Smith) *Let* G *be a finite group with* $m = \mathrm{rk}(G)$*. If the Euler class associated to some map* $\varphi : BG \to BU(n)$ *is effective then* G *acts on a finite CW-complex* $X \simeq \mathbb{S}^{2n-1}$ *such that the isotropy groups have rank at most* $m - 1$*.*

Corollary 5 *Let G be a finite group with $2 = \mathrm{rk}(G)$. If the Euler class associated to some map $\varphi : BG \to BU(n)$ is effective then $h(G) = 2$.*

So we have reduced the problem to finding a particular type of map $\varphi : BG \to BU(n)$. We will recall two properties of such maps:

- If $\varphi, \psi : BG \to BU(n)$ are homotopic maps, then $\varphi^*(\xi) = \psi^*(\xi)$ in $H^{2n}(BG, \mathbb{Z})$ for any $\xi \in H^{2n}(BU(n), \mathbb{Z})$.

- If P is a p-group then any map $\varphi : BP \to BU(n)$ is homotopic to $B\lambda$ for some unitary representation $\lambda : P \to U(n)$.

G_p will denote a Sylow p-subgroup of G.

$\mathrm{Char}_n(G_p)$ will be the set of degree n complex characters of G_p

$\mathrm{Char}_n^G(G_p)$ will be the subset of $\mathrm{Char}_n(G_p)$ consisting of those degree n complex characters of G_p that are the restrictions of class functions on G (i.e. that respect the fusion in G).

Recall that $\mathrm{Rep}(G, U(n)) = \mathrm{Hom}(G, U(n))/\mathrm{Inn}(U(n))$ is the set of unitary representations of G.

Now we define a map $\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p)$, by using the following diagram:

$$\begin{array}{ccccc}
[BG, BU(n)] & \xrightarrow{\cong} & \prod_{p||G|}[BG, BU(n)^\wedge_p] & \xrightarrow{\text{res}} & \prod_{p||G|}[BG_p, BU(n)^\wedge_p] \\
\bar{\psi}_G\downarrow & & \bar{\phi}_G\downarrow & & \cong\uparrow \\
\prod_{p||G|}\mathsf{Char}_n(G_p) & = & \prod_{p||G|}\mathsf{Char}_n(G_p) & \xleftarrow{\cong} & \prod_{p||G|}\mathsf{Rep}(G_p, U(n)).
\end{array}$$

Notice that the image of $\bar{\psi}_G$ is contained in $\prod_{p||G|}\mathsf{Char}_n^G(G_p)$. So we let ψ_G be the resulting map from $[BG, BU(n)]$ to $\prod_{p||G|}\mathsf{Char}_n^G(G_p)$.

Theorem 6 (Jackson) *If G is a finite group of rank two, then the natural mapping*

$$\psi_G : [BG, BU(n)] \to \prod_{p||G|}\mathsf{Char}_n^G(G_p)$$

is a surjection.

Outline of Proof:

- Recall that a p-subgroup $P \subseteq G$ is said to be *principal p-radical* if $Z(P)$ is a Sylow p subgroup of $C_G(P)$ and $N_G(P)/PC_G(P)$ is p-reduced.

- Using obstruction theory we see that if for each prime p dividing $|G|$ and principal p-radical subgroup $P \subseteq G$, $\mathrm{rk}_p(N_G(P)/P) \leq 2$, then the map ψ_G is a surjection.

- The remaining step is to show that the only subgroups which can appear as non-Sylow, principal p-radical subgroups of rank two groups G would have the property that $\mathrm{rk}_p(N_G(P)/P) \leq 2$.

Definition 7 *Let G be a finite group, p a prime dividing $|G|$, and G_p a Sylow p-subgroup of G. A character χ of G_p is called a p-*effective character of G *if $\chi \in \mathrm{Char}_n^G(G_p)$ and for each $E \subseteq G_p$ elementary abelian with $\mathrm{rk}(E) = \mathrm{rk}(G)$, $[\chi|_E, 1_E] = 0$.*

Theorem 8 (Jackson) *Let G be a finite group. If for each prime p dividing $|G|$ there exists a p-effective character of G, then there is a map $\varphi : BG \to BU(n)$ whose associated Euler class is effective.*

Corollary 9 *Let G be a finite group with* $\mathrm{rk}(G) = 2$. *If for each prime p dividing the order of G there exists a p-effective character of G, then G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{N_1} \times \mathbb{S}^{N_2}$.*

A definition from group theory will be necessary in demonstrating the existence of p-effective characters.

Definition 10 *Let G be a finite group and H and K subgroups such that $H \subset K$. We say that H is* strongly closed in K with respect to G *if for each $g \in G$, $H^g \cap K \subseteq H$.*

We now state a useful sufficient condition for the existence of a p-effective character.

Proposition 11 *Let G be a finite group,* $\mathrm{rk}(G) = n$, *p a prime divisor of $|G|$, and G_p a Sylow p-subgroup of G. If there exists $H \subseteq Z(G_p)$ such that H is non-trivial and strongly closed in G_p with respect to G, then G has a p-effective character.*

We next define a group which does not have a p-effective character.

Fix an odd prime p.
Let $T_p \cong (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes_\theta \mathrm{SL}_2(\mathbb{F}_p)$ where the action θ is given by the obvious inclusion $\mathrm{SL}_2(\mathbb{F}_p) \to \mathrm{GL}_2(\mathbb{F}_p) \cong \mathrm{Aut}(\mathbb{Z}_p \times \mathbb{Z}_p)$.

Notice that for p an odd prime, $T_p \subseteq PSL_3(\mathbb{F}_p)$.

Theorem 12 (Jackson) *Let G be a finite group and $p > 2$ a prime with $\mathrm{rk}_p(G) = \mathrm{rk}(G) = 2$ and let $G_p \in \mathrm{Syl}_p(G)$. If $\Omega_1(Z(G_p))$ is not strongly closed in G_p with respect to G, then G has a subgroup H such that $H/O_{p'}(H) \cong T_p$.*

Outline of proof:

- Since G is rank 2 it is clear that $Z(G)$ must be cyclic. And so $\Omega_1(Z(G_p))$ is cyclic of order p.

- Using Alperin's Fusion Theory, we know that there exists a proper principal p-radical subgroup P of G_p such that $\Omega_1(Z(G_p))$ is not strongly closed in P with respect to $N_G(P)$.

- By the classification of rank two p-groups for odd primes we see that P must be metacyclic, since $Z(P)$ must have rank two.

- By work of Diaz, Ruiz, and Viruel, such a P must by homocyclic abelian.
  (In fact, either $p = 3$ or P has exponent p.)

- $\Omega_1(P)$ becomes the rank two elementary abelian in T_p, and since P is principal p-radical $N_G(P)/PC_G(P)$ must contain $SL_2(\mathbb{F}_p)$.

- G must contain a subgroup H such that $H/O_{p'}(H) \cong T_p$.

Now we will look at the prime two.

We begin with a restatement of Proposition 7.1 of Alperin, Brauer, and Gorenstein, "Finite simple groups of 2-rank two".

Theorem 13 (Alperin, Brauer, and Gorenstein) *Let G be a finite group such that $\mathrm{rk}_2(G) = 2$ and let $G_2 \in \mathrm{Syl}_2(G)$. If $\Omega_1(Z(G_2))$ is not strongly closed in G_2 with respect to G, then G_2 is either dihedral, semi-dihedral, or wreathed.*

Theorem 14 (Jackson) *If G is a finite group with a dihedral, semi-dihedral, or wreathed Sylow 2-subgroup such that $\mathrm{rk}(G) = 2$, then G has a 2-effective character.*

Corollary 15 *For any rank two finite group G, G has a 2-effective character.*

So putting together this result for the prime two and the previous results for odd primes, we have shown the following:

Theorem 16 (Jackson) *Let G be a finite group such that* $\mathrm{rk}(G) = 2$. *G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ unless for some odd prime p, G contains a subgroup H such that $H/O_{p'}(H) \cong T_p$.*

Conclusions:

It has been shown by O. Unlu (2004) that T_p for an odd prime p cannot act on a homotopy sphere with rank one isotropy groups.

So the question of whether such groups have homotopy rank two, cannot be approach using the methods developed by Adem and Smith.

It is unknown if T_p for an odd prime p (and any group containing a subgroup H such that $H/O_{p'}(H) \cong T_p$) has homotopy rank two.

Group Theory Definitions:

Definition 17 *Recall that a 2-group is* semi-dihedral *(sometimes called quasi-dihedral) if it is generated by two elements* x *and* y *subject to the relations that* $y^2 = x^{2^n} = 1$ *and* $yxy^{-1} = x^{-1+2^{n-1}}$ *for some* $n \geq 3$*.*

Definition 18 *A 2-group is called* wreathed *if it is generated by three elements* x*,* y*, and* z *subject to the relations that* $x^{2^n} = y^{2^n} = z^2 = 1$*,* $xy = yx$*, and* $zxz^{-1} = y$ *with* $n \geq 2$*.*