A CLASSIFICATION OF 2-LOCAL FINITE GROUPS OF RANK TWO WITH TRIVIAL CENTER

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Abstract. A classification of 2-local finite groups of rank two that have a trivial center is demonstrated. This work uses known results on the 2-fusion of finite simple groups of rank two translated into the language of p-local finite groups and saturated fusion systems. This classification of certain 2-local finite groups extends a recent result of Diaz, Ruiz, and Viruel. In their work, they classify all p-local finite groups of rank two for odd primes p. No claim is made that this is a new result. It serves more as a collection of the necessary results to classify 2-local finite groups of rank two that have a trivial center.

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1. Introduction

The purpose of this paper is to classify all 2-local finite groups of rank two that have a trivial center. The concept of p-local finite groups have been defined by Broto, Levi, and Oliver [7] in their study of p-local homotopy theory of finite groups. p-local finite groups are purely algebraic objects, which generalize the p-structure of finite groups: every finite group gives rise to a p-local finite group. Several examples of exotic p-local finite groups, which do not come from any finite group, are known (see [7, Section 9], [9], [19], and [21]). Diaz, Ruiz, and Viruel have begun a systematic classification of p-local finite groups by classifying all p-local finite groups of rank two for odd primes [9, 21]. Theorem 1.1 contains a classification of all saturated fusion systems over rank two 2-groups such that the fusion systems have trivial center. In light of Remark 3.3, Theorem 1.1, in fact, classifies a 2-local finite groups of rank two with trivial center. For the relevant definitions refer to Sections 2 and 3.

Theorem 1.1. *Let S be a rank two 2-group and let $\mathcal{F}$ be a saturated fusion system over S. If $\mathcal{F}$ has a trivial center, then S is either homocyclic abelian, dihedral, semi-dihedral,* *wreathed, or isomophic to a Sylow 2-subgroup of* $PSU_3(\mathbb{F}_4)$ *and in addition one of the following statements holds for* $\mathcal{F}$*:*

(1) $\mathcal{F}$ *has no proper* $\mathcal{F}$*-Alperin subgroups and* $\mathcal{F}$ *is the fusion system of a finite group* $S : G$ *where* G *is a odd order subgroup of* $\mathrm{Out}(S)$*.*
(2) $\mathcal{F}$ *is the fusion system of the finite group* $(\mathbb{Z}_2 \times \mathbb{Z}_2) : \Sigma_3$*.*
(3) $\mathcal{F}$ *is the fusion system of one of the following finite groups for a prime* $p \geq 5$*:* $PSL_3(\mathbb{F}_p)$*,* $PSL_2(\mathbb{F}_p)$*, or* $PGL_2(\mathbb{F}_p)$*.*
(4) $\mathcal{F}$ *is the fusion system of an extension of* $PSL_2(\mathbb{F}_{q^2})$ *by a cyclic group of order 2 for some appropriate odd prime* q*.*

Proof. The classification of the 2-group S is a result of Theorem 5.2. If S is homocyclic abelian or isomorphic to a Sylow 2-subgroup of $PSU_3(\mathbb{F}_4)$, then we see that the first statement holds by Corollary 4.6 and Proposition 5.9, respectively. The nontrivial fusion systems over dihedral 2-groups are classified in Theorem 5.12, where each such fusion system is shown to be the fusion system of one of the groups in statement three. The three non-trivial fusion systems over a wreathed 2-group are classified in Theorem 5.14. One fusion system has a non-trivial center, another is the fusion system describe in the second statement, and the final fusion system is the fusion system of $PSL_3(\mathbb{F}_p)$ in which case the second statement holds when S is a wreathed 2-group. A classification of the three non-trivial fusion systems over a semi-dihedral 2-group is contained in Theorem 5.16. One fusion system in Theorem 5.16 has a non-trivial center, while for another fusion system statement three holds. The third fusion system in Theorem 5.16 is the one described in statement four. $\square$

The classification of p-local finite groups of rank two for odd primes given by Diaz, Ruiz, and Viruel depends on first having a classification of all p-groups of rank two for odd primes (see [5] and [9, Appendix]). There is no known similar classification of all rank two 2-groups although partial classifications have been found [16, 17, 18, 22]. Because of the lack of a classification of all rank two 2-groups, realizing a classification of all 2-local finite groups of rank two has proven elusive. For this reason, we only classify here those 2-local finite groups of rank two that have trivial center.

Besides their intrinsic interest as algebraic objects, p-local finite groups relate to other areas of algebra and topology by encoding possible p-local structures for finite groups. For example, p-local structures of finite groups are used when constructing actions on spheres and on homotopy spheres [1, 2, 15]. In addition, a milestone in the proof of The Classification of Finite Simple Groups [13] is the classification of finite simple groups of 2-rank two (see [10, Chapter 1]). This classification of such finite simple groups depends on knowing the 2-fusion of finite simple groups of 2-rank two (see [11, Theorem 7.7.3] and [10, Prop. 1.64]) and thus is closely related to the present classification of rank two 2-local finite groups (e.g. see Theorem 5.2).

2. Definitions of p-local finite groups

Building on previously unpublished work of L. Puig, the basic definitions were introduced in [7]. For additional information and a survey on p-local finite groups see [8]. For our puposes here, we begin to define p-local finite groups by explaining fusion systems.

Definition 2.1. A fusion system $\mathcal{F}$ over a finite p-group S is a category consisting of both objects that subgroups of S, and morphisms that are sets $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ and satisfy the following two conditions:

(a) $\mathrm{Hom}_S(P, Q) \subseteq \mathrm{Hom}_{\mathcal{F}}(P, Q) \subseteq \mathrm{Inj}(P, Q)$ for all P and Q subgroups of S, and

(b) every morphism in $\mathcal{F}$ factors as an isomorphism in $\mathcal{F}$ followed by an inclusion.

We say that two subgroups $P, Q \subseteq S$ are *$\mathcal{F}$-conjugate* if $\mathrm{Hom}_{\mathcal{F}}(P, Q)$ is non-empty and $|P| = |Q|$, meaning that there is an isomophism in $\mathcal{F}$ between P and Q. $\mathrm{Hom}_{\mathcal{F}}(P, P)$ is denoted by $\mathrm{Aut}_{\mathcal{F}}(P)$, and $\mathrm{Out}_{\mathcal{F}}(P)$ is used to mean $\mathrm{Aut}_{\mathcal{F}}(P)/\mathrm{Aut}_P(P)$ for our purposes. We, then, define two special types of subgroups that occur in fusion systems. These subgroups are subsequently used to define saturated fusion systems.

Definition 2.2. Let $\mathcal{F}$ be a fusion system over a p-group S.
(1) A subgroup $P \subseteq S$ is *fully centralized in* $\mathcal{F}$ if $|C_S(P)| \geq |C_S(P')|$ for every $P' \subseteq S$ that is $\mathcal{F}$-conjugate to P.
(2) A subgroup $P \subseteq S$ is *fully normalized in* $\mathcal{F}$ if $|N_S(P)| \geq |N_S(P')|$ for every $P' \subseteq S$ that is $\mathcal{F}$-conjugate to P.

Definition 2.3. A fusion system $\mathcal{F}$ over a p-group S is *saturated* if the following two conditions hold:
(1) For each fully normalized subgroup P in $\mathcal{F}$, P is fully centralized, and $\mathrm{Aut}_S(P) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}}(P))$.
(2) If $P \subseteq S$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, S)$ are such that $\varphi(P)$ is fully centralized, and if we set $N_\varphi = \{g \in N_S(P) | \varphi c_g \varphi^{-1} \in \mathrm{Aut}_S(\varphi(P))\}$, then there is a $\bar{\varphi} \in \mathrm{Hom}_{\mathcal{F}}(N_\varphi, S)$ such that $\bar{\varphi}|_P = \varphi$.

Remark 2.4 (See [7, Proposition 1.3])**.** Every finite group G gives rise to a saturated fusion system. Picking $S \in \mathrm{Syl}_p(G)$, the objects of $\mathcal{F}_S(G)$ are the subgroups of S while the morphism sets are $\mathrm{Hom}_{\mathcal{F}_S(G)}(P, Q) = \mathrm{Hom}_G(P, Q)$.

As saturated fusion systems generalize the theory of finite groups, some classical results in group theory can also be generalize to saturated fusion systems. One such example, which will prove useful here, is Alperin's fusion theorem.

Definition 2.5. Let $\mathcal{F}$ be a fusion system over a p-group S.
(a) A subgroup $P \subseteq S$ is called $\mathcal{F}$*-centric* if P and all its $\mathcal{F}$-conjugates contain their S-centralizers.
(b) A subgroup $P \subseteq S$ is called $\mathcal{F}$*-radical* if $\mathrm{Out}_{\mathcal{F}}(P)$ is p-reduced: if $\mathrm{Out}_{\mathcal{F}}(P)$ has not nontrivial normal p-subgroups

Theorem 2.6 (Alperin's fusion theorem for saturated fusion systems, See [7, Theorem A. 10])**.** *Let $\mathcal{F}$ be a saturated fusion system over a p-group S. For each morphism $\psi \in$*

$\mathrm{Aut}_{\mathcal{F}}(P, P')$, *there exist two sequences of subgroups of* S*:* $P = P_0, P_1, \ldots, P_k = P'$ *and* $Q_1, Q_2, \ldots, Q_k$*, and morphisms* $\psi_i \in \mathrm{Aut}_{\mathcal{F}}(Q_i)$*, such that*
(1) Q_i *is fully normalized in* $\mathcal{F}$*,* $\mathcal{F}$*-centric, and* $\mathcal{F}$*-radical for each* i*;*
(2) $P_{i-1}, P_i \subseteq Q_i$ *and* $\psi_i(P_{i-1}) = P_i$ *for each* i*; and*
(3) $\psi = \psi_k \circ \psi_{k-1} \circ \cdots \circ \psi_1$.

The subgroups Q_i in Theorem 2.6 completely determine the structure of $\mathcal{F}$. This leads us to name such subgroups.

Definition 2.7. Let $\mathcal{F}$ be a fusion system over a p-group S. A subgroup $P \subseteq S$ is called $\mathcal{F}$-*Alperin* if it is fully normalized in $\mathcal{F}$, $\mathcal{F}$-centric, and $\mathcal{F}$-radical.

From Definition 2.7, it is clear that S is always an $\mathcal{F}$-Alperin subgroup for any saturated fusion system over S. The most important property of $\mathcal{F}$-Alperin subgroups follows from Alperin's fusion system for saturated fusion systems and is described in the following remark:

Remark 2.8. By Theorem 2.6, each morphism in a saturated fusion system $\mathcal{F}$ is the composition of restrictions of morphisms between $\mathcal{F}$-Alperin subgroups.

In order to define a p-local finite group, we need to define a centric linking system. In our notation, we let $\mathcal{F}^c$ be the full subcategory of $\mathcal{F}$ whose objects are the $\mathcal{F}$-centric subgroups of S.

Definition 2.9. Let $\mathcal{F}$ be a fusion system over the p-group S. A *centric linking system associated to* $\mathcal{F}$ is a category $\mathcal{L}$ whose objects are the $\mathcal{F}$-centric subgroups of S, together with a functor $\pi : \mathcal{L} \to \mathcal{F}^c$ and "distinguished" monomorphisms $\delta_P : P \to \mathrm{Aut}_{\mathcal{L}}(P)$ for each $\mathcal{F}$-centric subgroup $P \subseteq S$, which satisfy the following conditions:
(A) π is the identity on the objects and surjective on the morphisms. More precisely, for each pair of objects $P, Q \in \mathcal{L}$, $Z(P)$ acts freely on $\mathrm{Mor}_{\mathcal{L}}(P, Q)$ by composition–upon identifying $Z(P)$ with $\delta_P(Z(P)) \subseteq \mathrm{Aut}_{\mathcal{L}}(P)$–, and π induces a bijection

$$\mathrm{Mor}_{\mathcal{L}}(P, Q)/Z(P) \to \mathrm{Hom}_{\mathcal{F}}(P, Q).$$

(B) For each $\mathcal{F}$-centric subgroup $P \subseteq S$ for each $g \in P$, π sends $\delta_P(g) \in \mathrm{Aut}_{\mathcal{L}}(P)$ to $c_g \in \mathrm{Aut}_{\mathcal{F}}(P)$.

(C) For each $f \in \mathrm{Mor}_{\mathcal{L}}(P, Q)$ and each $g \in P$, the equality $\delta_Q(\pi(f)(g)) \circ f = f \circ \delta_P(g)$ holds in $\mathcal{L}$.

Definition 2.10. A *p-local finite group* is a triple $(S, \mathcal{F}, \mathcal{L})$ where S is a p-group, $\mathcal{F}$ is a saturated fusion system over S, and $\mathcal{L}$ is a centric linking system associated to $\mathcal{F}$. The classifying space of the p-local finite group $(S, \mathcal{F}, \mathcal{L})$ is the space $|L|_p^\wedge$.

Now that we have defined p-local finite groups, in the next section we will review some additional theory that will be necessary in proving the classification given in Theorem 1.1.

3. The Theory of p-Local Finite Groups

For a given fusion system $\mathcal{F}$ over a p-group S, there is an obstruction theory for the existence and uniqueness of a centric linking system associated to $\mathcal{F}$ and a p-local finite group with p-group S and fusion system $\mathcal{F}$. For a discussion of this particular obstruction theory, see Section 3 of [7]. For fusion systems over a p-group of small rank, the obstructions can be shown to be trivial.

Theorem 3.1 ([7, Corollary 3.4])**.** *Let $\mathcal{F}$ be a fusion system over a p-group S. If $\mathrm{rk}_p(S) < p^3$, then there exists a centric linking system associated to $\mathcal{F}$; furthermore if $\mathrm{rk}_p(S) < p^2$, then the centric linking system associated to $\mathcal{F}$ is unique.*

Because a p-local finite group of rank two has a saturated fusion system over a rank two p-group, Corollary 3.2 simplifies the situation in the rank two case.

Corollary 3.2 ([9])**.** *Let p be a prime. The set of p-local finite groups over a rank two p-group S is in bijective correspondence with the set of saturated fusion systems over S.*

Remark 3.3. Classifying 2-local finite groups of rank two is a matter of classifying fusion systems over rank two 2-groups; the classification in Theorem 1.1 is in fact a classification of all 2-local finite groups of rank two with trivial center.

In Definitions 3.4 and 3.5 induced fusion systems are explained over fully centralized subgroups and fully normalized subgroups, respectively. Definition 3.4 will subsequently be used to define the center of a fusion system while Definition 3.5 will be used in Section 4 to define resistant p-groups.

Definition 3.4 ([7, Section 2])**.** Let $\mathcal{F}$ be a fusion system over S and $P \subseteq S$ a fully centralized subgroup in $\mathcal{F}$. The *centralizer fusion system of* P *in* $\mathcal{F}$, $C_{\mathcal{F}}(P)$ is the fusion system over $C_S(P)$ with objects $Q \subseteq C_S(P)$ and morphisms

$$\mathrm{Hom}_{C_{\mathcal{F}}(P)}(Q, Q') = \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, Q') | \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(QP, Q'P), \psi|_Q = \varphi, \psi|_P = \mathrm{Id}_P\}.$$

Definition 3.5 ([7, Section 6])**.** Let $\mathcal{F}$ be a fusion system over S and $P \subseteq S$ a fully normalized subgroup in $\mathcal{F}$. The *normalizer fusion system of* P *in* $\mathcal{F}$, $N_{\mathcal{F}}(P)$ is the fusion system over $N_S(P)$ with objects $Q \subseteq N_S(P)$ and morphisms

$$\mathrm{Hom}_{N_{\mathcal{F}}(P)}(Q, Q') = \{\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, Q') | \exists \psi \in \mathrm{Hom}_{\mathcal{F}}(QP, Q'P), \psi|_Q = \varphi, \psi|_P \in \mathrm{Aut}(P)\}.$$

Definition 3.6. Let $\mathcal{F}$ be a fusion system over S. A subgroup $P \subseteq Z(S)$ is called *central in* $\mathcal{F}$ if the centalizer fusion system $C_{\mathcal{F}}(P)$ is $\mathcal{F}$.

Proposition 3.7 ([6, Proposition 6.1])**.** *Let* $\mathcal{F}$ *be a saturated fusion system over* S. *Let* $Z_{\mathcal{F}}(S) = \{x \in S | \varphi(x) = x \ \forall \varphi \in Mor(\mathcal{F}^c)\}$. $Z_{\mathcal{F}}(S)$ *is a subgroup of* $Z(S)$*, which is central in* $\mathcal{F}$*, and contains every central subgroup of* $\mathcal{F}$*. We call* $Z_{\mathcal{F}}(S)$ *the center of* $\mathcal{F}$*.*

Remark 3.8. By Remark 2.8, $Z_{\mathcal{F}}(S) = \{x \in S | \varphi(x) = x \ \forall \varphi \in \mathrm{Mor}(\mathcal{F})\}$. This characterization of $Z_{\mathcal{F}}(S)$ will be used to show that a given fusion system has no center.

Next, we describe the case where S is the only $\mathcal{F}$-Alperin subgroup of S.

Remark 3.9 ([9, Lemma 2.15])**.** Let $\mathcal{F}$ be a saturated fusion system over a p-group S. $\mathcal{F} = N_{\mathcal{F}}(S)$ if and only if S is itself the only $\mathcal{F}$-Alperin subgroup of S. In this case, $\mathcal{F}$ is the fusion system associated to the finite group $S : \mathrm{Out}_{\mathcal{F}}(S)$.

In classifying saturated fusion systems over a given p-group, we need to be able to construct fusion systems which are saturated. One way to build fusion systems over a p-group S is to make changes to a fusion system of an extension of the p-groups S. Proposition 3.12 gives sufficient conditions for a fusion system constructed in such a way to be saturated. We start with a definition.

Definition 3.10. Let S be a p-group. A subgroup $P \subseteq S$ is called p-centric if $C_S(P) = Z(P)$.

Remark 3.11. Let $\mathcal{F}$ be a fusion system over a p-group S. If a subgroup $P \subseteq S$ is an $\mathcal{F}$-centric subgroup, then P is a p-centric subgroup.

Proposition 3.12 ([8, Proposition 5.3])**.** *Fix a finite group G, a sylow p-subgroup S, and subgroups $Q_1, \dots, Q_m \subseteq S$ such that no Q_1 is conjugate to a subgroup Q_j for $i \neq j$. Set $K_i = \mathrm{Out}_G(Q_i)$, and fix subgroups $\Delta_i \subseteq \mathrm{Out}(Q_i)$ that contain K_i. Assume that for each i*
1) $p \not| [\Delta_i : K_i]$;
2) Q_i is p-centric in G, but for each $P \subsetneq Q_i$ there is $\alpha \in \Delta_i$ such that $\alpha(P)$ is not p-centric in G; and
3) for all $\alpha \in \Delta_i \setminus K_i$, $K_i \cap \alpha K_i \alpha^{-1}$ has order prime to p.
Then the fusion system $\mathcal{F} = \langle \mathcal{F}_S(G), \mathcal{F}_{Q_1}(\Delta_1), \dots, \mathcal{F}_{Q_1}(\Delta_1) \rangle$ is saturated and has an associated centric linking system.

We will make use of Proposition 3.12, by means of Corollary 3.13 to show that certain fusion systems are saturated.

Corollary 3.13 (See [9, saturation portion of proof of Theorem 5.9])**.** *Let $\mathcal{F}$ be a fusion system over S. Define $G = S : \mathrm{Out}_{\mathcal{F}}(S)$. For each G-conjugacy class of $\mathcal{F}$-Alperin subgroups, let $P \subseteq S$ be a representative and set $K_P = \mathrm{Out}_G(P)$ and $\Delta_P = \mathrm{Out}_{\mathcal{F}}(P)$. $\mathcal{F}$ is a saturated fusion system if for each $\mathcal{F}$-Alperin representative P the following hold:*
1) P does not contain any proper $\mathcal{F}$-centric subgroups, and
2) $p \not| [\Delta_P : K_P]$ and for each $\alpha \in \Delta_P \setminus K_P$, $K_P \cap \alpha^{-1} K_P \alpha$ has order prime to p.

We end this section with a discussion of fusion systems that contain a nontrivial center. As we classify 2-local finite groups of rank two with trivial center, we will also arrive at some 2-local finite groups with non-trivial center. We will want to demonstrate that these 2-local finite groups are not exotic. This will be done using Lemma 3.16.

Definition 3.14. Let $\mathcal{F}$ be a fusion system over a finite p-group S and let $A \subseteq Z(S)$ be a central subgroup in $\mathcal{F}$. Define $\mathcal{F}/A$ to be the fusion system over S/A with morphism sets given by

$$\mathrm{Hom}_{\mathcal{F}/A}(P/A, Q/Q) = \mathrm{Im}[\mathrm{Hom}_{\mathcal{F}}(P, Q) \to \mathrm{Hom}(P/A, Q/Q)].$$

Lemma 3.15 ([7, Lemma 5.6])**.** *If $\mathcal{F}$ is a saturated fusion system over a finite p-group S and $A \subseteq Z_{\mathcal{F}}(S)$, then $\mathcal{F}/A$ is a saturated fusion system over S/A.*

Lemma 3.16 ([6, Corollary 6.17])**.** *Let $\mathcal{F}$ be a saturated fusion system over a finite p-group S and assume that there is a nontrivial subgroup $A \subseteq Z(S)$ that is central in $\mathcal{F}$. $\mathcal{F}$ is the fusion system of a finite group if and only if $\mathcal{F}/A$ is the fusion system of a finite group.*

4. Resistant p-Groups

In Section 3, we revisted most of the relevant theory concerning saturated fusion systems. We will be classifying fusion systems over a given p-group; therefore, it will be important to know when every saturated fusion system $\mathcal{F}$ over a given p-group has only one $\mathcal{F}$-Alperin subgroup. This leads to the following definition.

Definition 4.1. A p-group S is called *resistant* if for any saturated fusion system $\mathcal{F}$ over S, $N_{\mathcal{F}}(S) = \mathcal{F}$.

Resistant p-groups are the generalization of Swan groups for fusion systems. By Remark 3.9, if S is a resistant p-group, for any saturated fusion system $\mathcal{F}$ over S, S is the only $\mathcal{F}$-Alperin subgroup.

Remark 4.2. Fixing a prime p, a finite group G, and $S \in \mathrm{Syl}_p(G)$, a subgroup $H \subseteq G$ is said to *control p-fusion in G* if S is contained in H and for each $P \subseteq S$ with $P, gPg^{-1} \subseteq H$, $g = hc$ with $h \in H$ and $c \in C_G(P)$.

A p-group S is called a Swan group if for any finite group G with $S \in \mathrm{Syl}_p(G)$, $N_G(S)$ controls p-fusion in G. Every resistant p-group S is a Swan group because the fusion system associated to any finite group G such that $S \in \mathrm{Syl}_p(G)$ is also the fusion system of $N_G(S)$. Now we consider sufficient conditions for a p-group S to be resistant. Notice that if S is a resistant p-group, by Remark 3.9, S is the only $\mathcal{F}$-Alperin subgroup for any saturated fusion system $\mathcal{F}$. Diaz, Ruiz, and Viruel have found such sufficient conditions following the ideas of [20, Section 2]. Theorem 4.3 gives a sufficient condition for a p-centric subgroup to not be $\mathcal{F}$-Alperin in any fusion system $\mathcal{F}$. We follow with a corollary giving a simplified version of this sufficient condition.

Theorem 4.3 (See [9, Theorem 3.3])**.** *Let S be a p-group. Any p-centric subgroup $P \subseteq S$ such that $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}(P)) \neq 1$ is not $\mathcal{F}$-Alperin for any fusion system $\mathcal{F}$ over S.*

Corollary 4.4. *Let S be a p-group. If $P \subsetneq S$ is a p-centric subgroup such that $\mathrm{Out}(P)$ is a p-group, then P is not $\mathcal{F}$-Alperin for any fusion system $\mathcal{F}$ over S.*

Proof. Suppose that $P \subsetneq S$ is p-centric with $\mathrm{Out}(P)$ a p-group. Since P is p-centric $C_S(P) \subseteq P$, but $P \neq S$, $P \subsetneq N_S(P)$; thus, $\mathrm{Out}_S(P) \neq 1$. Since $\mathrm{Out}(P)$ is a p-group, $O_p(\mathrm{Out}(P)) = \mathrm{Out}(P)$; therefore, $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}(P)) \neq 1$. By Theorem 4.3, P is not $\mathcal{F}$-Alperin for any fusion system $\mathcal{F}$ over S. $\square$

Corollary 4.5 uses Theorem 4.3 to give sufficient conditions for a p-group S to be resistant.

Corollary 4.5 ([9, Theorem 3.3])**.** *Let S be a p-group. If every proper p-centric subgroup $P \subseteq S$ satisfies $\mathrm{Out}_S(P) \cap O_p(\mathrm{Out}(P)) \neq 1$, then S is a resistant group.*

We conclude this section by giving a family of resistant groups which will be used in the classification of 2-local finite groups of rank two with trivial center.

Corollary 4.6 ([9, Corollary 3.4]). *Abelian p-groups are resistant groups.*

5. 2-Local Finite Groups with Trivial Center

In this section, we prove the theorems used in proving the classification in Theorem 1.1. We begin by giving another characterization of involutions contained in the center of a fusion system.

Lemma 5.1. *Let $\mathcal{F}$ be a fusion system over a finite 2-group S. An involution $x \in S$ is in the center of $\mathcal{F}$ if and only if $\langle x \rangle$ is strongly closed in $\mathcal{F}$.*

Proof. Suppose $x \in Z_{\mathcal{F}}(S)$. So for all $\varphi \in \mathrm{Mor}(\mathcal{F}, \varphi(x) = x$; therefore x is not $\mathcal{F}$-conjugate to any other element of S and $\langle x \rangle$ is strongly closed in $\mathcal{F}$.
Now suppose that $\langle x \rangle$ is strongly closed in $\mathcal{F}$. Clearly x is not $\mathcal{F}$-conjugate to any other element of S. So for all $\varphi \in \mathrm{Mor}(\mathcal{F})$, $\varphi(x) = x$; thus, $x \in Z_{\mathcal{F}}(S)$. □

Theorem 5.2 characterizes the types of 2-groups S which can have a center free fusion sytem over S. The proof of Theorem 5.2 follows from the proof of a theorem by Alperin, Brauer, and Gorenstein [4, Proposition 7.1].

Theorem 5.2 ([4, Proposition 7.1 and its proof]). *Let $\mathcal{F}$ be a fusion system over a finite 2-group S of rank two. If $Z_{\mathcal{F}}(S)$ is trivial, then S is either homocyclic abelian, dihedral, semi-dihedral, wreathed, or $S \cong U_{64} \in \mathrm{Syl}_2(PSL_3(\mathbb{F}_4))$.*

Since this proof follows from the proof of Proposition 7.1 of [4], we will only state a brief outline, including the changes needed to prove our Theorem 5.2.
If S does not contain a normal rank 2 elementary abelian subgroup, then S must be either dihedral or semi-dihedral (see [11, Theorem 5.4.10]). We may assume that S has a normal rank 2 elementary abelian subgroup, which we will call V. Letting $T = C_S(V)$, we notice that $[S : T] \leq 2$.

Consider the case $|S : T| = 1$. So $V \subseteq Z(S)$ and the three order two elements of V are $\mathcal{F}$-conjugate. So S is a rank two Suzuki 2-group; therefore, $S \cong U_{64}$ or S is homocyclic abelian [14].

Consider now the case $[S : T] = 2$.

We notice that $\mathrm{Out}_{\mathcal{F}}(V)$ is isomorphic to a subgroup of $\mathrm{Out}(V) \cong \Sigma_3$ and that $2\big| \, |\mathrm{Out}_{\mathcal{F}}(V)|$. Thus either $\mathrm{Out}_{\mathcal{F}}(V) \cong \Sigma_3$ or $2 = |\mathrm{Out}_{\mathcal{F}}(V)|$.

From the argument in [4] we get the following results: if $\mathrm{Out}_{\mathcal{F}}(V) \cong \Sigma_3$, then S is wreathed (including the case of D_8); and if $|\mathrm{Out}_{\mathcal{F}}(V)| = 2$, then $S \cong D_8$. This finishes the discussion of Theorem 5.2.

Next we give necessary conditions for a subgroup of a 2-group S to be $\mathcal{F}$-Alperin in some fusion system over S.

Lemma 5.3. *Let $\mathcal{F}$ be a fusion system over a 2-group S. If P is a proper $\mathcal{F}$-Alperin subgroup of S, then P is 2-centric subgroup of S and P has an automorphism of odd order.*

Proof. P is p-centric by Remark 3.11. By Corollary 4.4, $\mathrm{Out}(P)$ cannot be a 2-group. □

By Lemma 5.3, 2-centric subgroups which have an odd order automorphism will play an important roll in the theory. This leads us to make the following definition:

Definition 5.4. Let S be a 2-group. We say that a proper subset $P \subsetneq S$ is 2-*Alperin* if P is 2-centric and P has an automorphism of odd order.

Remark 5.5. If $\mathcal{F}$ is a fusion system over a 2-group S. Clearly by Lemma 5.3, every proper $\mathcal{F}$-Alperin subgroup of S is a 2-Alperin subgroup of S.

Richard Thomas has classified all 2-groups of small rank which have an automorphism of odd order [23, 24].

Theorem 5.6 (Richard Thomas [23, 24])**.** *Let P be a 2-group of rank one or two. If P has an automorphism of odd order, then one of the following holds:*

(1) $P \cong Q_8 * D_8$,

(2) $P \cong Q_8 \times Q_8$,

(3) $P \cong Q_8 \wr \mathbb{Z}_2$,

(4) $P \cong Q_8 * C$ *where* $\mathrm{Out}(C)$ *is a* 2*-group,* $\mathrm{Out}(C) \lhd \mathrm{Out}(P)$*, and* $C \neq D_8$,

(5) $P \cong \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}$*, or*

(6) $P \cong U_{64} \in \mathrm{Syl}_2(\mathrm{PSU}_3(\mathbb{F}_4))$.

Our notation is as follows: Q_8 *means the quaternion 8 group,* D_8 *means the dihedral group of size 8,* $*$ *refers to the central product, and* $\wr$ *refers to the wreath product.*

Theorem 5.6 classifies all groups which can appear as 2-Alperin subgroups. We restate as the following corollary:

Corollary 5.7. *Let* P *be a proper subgroup of a* 2*-group* S*. If* P *is a* 2*-Alperin subgroup of* S*, then* P *is* p*-centric and is one of the groups listed in Theorem 5.6.*

Before we look at the specific possibilities for 2-groups S which could have a fusion system $\mathcal{F}$ over S with trivial center, we give sufficient conditions for such a fusion system to be saturated.

Lemma 5.8. *Let* $\mathcal{F}$ *be a fusion system over a* 2*-group* S *such that* $\mathrm{Out}_{\mathcal{F}}(S) = 1$*.* $\mathcal{F}$ *is a saturated fusion system if for each* $\mathcal{F}$*-Alperin subgroup* $P \subsetneq S$ *the following hold:*

(1) P *is abelian or* $P \cong Q_8 * \mathbb{Z}_{2^n}$ *(the central product of a quaternion eight group with a cyclic group),*

(2) $\mathrm{Out}_S(P) = \mathbb{Z}_2$*, and*

(3) $\mathrm{Out}_{\mathcal{F}}(P) = \Sigma_3$.

Proof. Using the notation of 3.13, $G = S : \mathrm{Out}_{\mathcal{F}}(S) = S$. Notice that any abelian subgroup group does not contain a proper p-centric subgroup; therefore for each $\mathcal{F}$-Alperin subgroup $P \subsetneq S$ which is abelian, P does not contain a proper $\mathcal{F}$-centric subgroup. Now assume that $P \subsetneq S$ is an $\mathcal{F}$-Alperin such that $P \cong Q_8 * Z(S)$. The only maximal subgroups of P which contain the center of P are three subgroups isomorphic to $\mathbb{Z}_4 * \mathbb{Z}_{2^n}$ (which is $\mathbb{Z}_2 \times \mathbb{Z}_{2^n}$ if $n > 1$ or $\mathbb{Z}_4$ if $P \cong Q_8$). Since these subgroups are abelian they are the only possible $\mathcal{F}$-centric subgroups of P. Notice that these abelian subgroups do not themselves have an odd order automorphism, none of them is $\mathcal{F}$-centric. Also for each $\mathcal{F}$-Alperin subgroup $P \subseteq S$, $K_P = \mathrm{Out}_G(P) = \mathbb{Z}_2$ and $\Delta_P = \mathrm{Out}_{\mathcal{F}}(P) = \Sigma_3$. Notice that $2 \not| [\Delta_P : K_P]$. Since $K_P = \mathbb{Z}_2$ is not normal in $\Delta_P = \Sigma_3$, for each $\alpha \in \Delta_P \setminus K_P$, $K_P \cap \alpha^{-1} K_P \alpha = 1$. The lemma follows from Corollary 3.13. □

Now we will start looking at the possibilities for S. Recall that if S is a homocyclic abelian 2-group, then S is a resistant group by Corollary 4.6. Next we consider a Sylow 2-subgroup of $\mathrm{PSU}_3(\mathbb{F}_4)$.

Proposition 5.9. *$U_{64} \in \mathrm{Syl}_2(\mathrm{PSU}_3(\mathbb{F}_4))$ is a resistant group.*

Proof: Let P be a proper 2-Alperin subgroup of U_{64}. We notice that $P \cong \mathbb{Z}_4 \times \mathbb{Z}_4$ and $P \lhd U_{64}$; therefore, $\mathrm{Out}_{U_{64}}(P) \cap O_p(\mathrm{Out}(P)) \neq 1$. So by Theorem 4.3, U_{64} is the only $\mathcal{F}$-Alperin subgroup of U_{64} for any fusion system $\mathcal{F}$ over U_{64}; therefore, U_{64} is resistant. □

This gives a new Swan group.

Corollary 5.10. *$U_{64} \in \mathrm{Syl}_2(\mathrm{PSU}_3(\mathbb{F}_4))$ is a Swan group.*

Before we look at the last three possibilities for S, the situation of a homocyclic abelian $\mathcal{F}$-Alperin proper subgroup must be examined.

Proposition 5.11 ([9, proof of Proposition 3.12])**.** *If $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ is $\mathcal{F}$-Alperin in some saturated fusion system $\mathcal{F}$ over S with $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} \subsetneq S$, then S contains a wreathed 2-group and $\mathrm{Aut}_{\mathcal{F}}(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}) \cong \Sigma_3$.*

Proof. Let $G = \mathrm{Aut}_{\mathcal{F}}(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n})$. Since $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ is $\mathcal{F}$-radical, G has no nontrivial normal 2-subgroup. We can consider $\mathrm{Aut}(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n})$ as 2×2 matrices with entries in $\mathbb{Z}_2^n$ and with odd determinant. Reduction modulo 2 induces the following short exact sequence where P is a p-subgroup:

$$\{1\} \to P \to \mathrm{Aut}(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}) \to \Sigma_3 \to \{1\}.$$

Restricting to G the short exact sequence becomes:

$$\{1\} \to P \cap G \to G \to H \to \{1\}$$

where H is a subgroup of Σ_3. $P \cap G$ must be trivial since it is a normal 2-subgroup of G; therefore, $G \cong H$. Since $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ is a proper subgroup of S, $2\big||G|$. G is not a 2-group, since $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ is $\mathcal{F}$-Alperin; thus, $G \cong \Sigma_3$. Clearly, $\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n} : \mathrm{Out}_S(\mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n})$ is a wreathed 2-group and a subgroup of S. □

Using Proposition 5.11, Theorem 5.12 classifies non-trivial saturated fusion systems over dihedral 2-groups.

Theorem 5.12. *Let $S = D_{2^n}$ be the dihedral group of order 2^n, $n \geq 3$. S has two conjugacy classes of rank two elementary abelian subgroups, which are permuted by an element of* $\mathrm{Out}(S)$. *If $\mathcal{F}$ is a nontrivial saturated fusion system over S, then* $\mathrm{Out}_{\mathcal{F}}(S) = 1$ *and either*
1) all rank two elementary abelian subgroups E of S are the only proper $\mathcal{F}$-Alperin subgroups of S and $\mathrm{Out}_{\mathcal{F}}(E) \cong \Sigma_3$*; or*
2) the rank two elementary abelian subgroups E_1 in one class are the only proper $\mathcal{F}$-Alperin subgroups of S, $\mathrm{Out}_{\mathcal{F}}(E_1) \cong \Sigma_3$*, and the rank two elementary abelian subgroups E_2 in the other class have* $\mathrm{Out}_{\mathcal{F}}(E_2) \cong \mathbb{Z}_2$.
In the first case, $\mathcal{F} = \mathcal{F}_S(PSL_2(\mathbb{F}_q))$ with q a prime such that $q \equiv 2^n \pm 1 (mod\ 2^{n+1})$. In the second case, $\mathcal{F} = \mathcal{F}_S(PGL_2(\mathbb{F}_q))$ with q a prime such that $q \equiv 2^{n-1} \pm 1 (mod\ 2^n)$.

Proof. Since S has no odd order automorphisms, $\mathrm{Out}_{\mathcal{F}}(S) = 1$. The rank two elementary abelian subgroups of S are the only 2-Alperin subgroups of S. By Proposition 5.11, the two cases in the theorem are the only possibilities. Both fusion systems listed are saturated, since they satisfy the hypotheses of Lemma 5.8. Realizing the the fusion system as fusion systems of finite groups follows from [11, Proposition 7.7.3]. □

In Proposition 5.11, $\mathrm{Out}_{\mathcal{F}}(P)$ was given for any homocyclic abelian proper $\mathcal{F}$-Alperin subgroup. Since a rank two elementary abelian subgroup is homocyclic abelian, we are left to do a similar analysis for another class of possible $\mathcal{F}$-Alperin subgroups. This analysis then will be use to classify non-trivial saturated fusion systems over wreathed and semi-dihedral 2-groups.

Lemma 5.13. *If $P = Q_8 * \mathbb{Z}_2^n$ (a central product of a quaternion eight group with a cyclic* 2*-group) is $\mathcal{F}$-Alperin in some saturated fusion system $\mathcal{F}$ over S with $P \subsetneq S$, then* $\mathrm{Out}_{\mathcal{F}}(P) \cong \Sigma_3$.

Proof. Since P is $\mathcal{F}$-radical, $\mathrm{Out}_{\mathcal{F}}(P)$ has no nontrivial normal 2-subgroup; thus, $\mathrm{Out}_{\mathcal{F}}(P) = \mathrm{Out}_{\mathcal{F}}(Q_8)$. Since $P \subsetneq S$, $2\big||\mathrm{Out}_{\mathcal{F}}(P)|$. Recalling that $\mathrm{Out}_{\mathcal{F}}(Q_8) \subseteq \mathrm{Out}(Q_8) = \Sigma_3$, it is clear that $\mathrm{Out}_{\mathcal{F}}(P) \cong \Sigma_3$. □

Theorem 5.14. *Let $S = \mathbb{Z}_{2^n} \wr \mathbb{Z}_2$. S has a normal subgroup $V = \mathbb{Z}_{2^n} \times \mathbb{Z}_{2^n}$ and one conjugacy class of subgroups isomorphic to $Q_8 * \mathbb{Z}_{2^n}$ which will be represented by Q. If $\mathcal{F}$ is a nontrivial saturated fusion system over S, then* $\mathrm{Out}_{\mathcal{F}}(S) = 1$ *and either*
1) V is the only proper $\mathcal{F}$-Alperin subgroup of S and $\mathrm{Out}_{\mathcal{F}}(V) \cong \Sigma_3$*;*
2) the subgroups Q are the only proper $\mathcal{F}$-Alperin subgroups of S and $\mathrm{Out}_{\mathcal{F}}(Q) \cong \Sigma_3$*; or*
3) the subgroups Q and the subgroups V are the only proper $\mathcal{F}$-Alperin subgroups of S, $\mathrm{Out}_{\mathcal{F}}(Q) \cong \Sigma_3$*, and* $\mathrm{Out}_{\mathcal{F}}(V) \cong \Sigma_3$*.*
In the first case $\mathcal{F} = \mathcal{F}_S(V : \Sigma_3)$ while the second case has a nontrivial center. In the third case $\mathcal{F}_S(PSL_3(\mathbb{F}_q))$ with q a prime such that $q \equiv 2^n + 1 (mod\ 2^{n+1})$.

Proof. Since $\mathrm{Out}(S)$ is a 2-group, $\mathrm{Out}_{\mathcal{F}}(S) = 1$. The only 2-Alperin subgroups of S are V and the subgroups Q. By Proposition 5.11 and Lemma 5.13, the three cases in the theorem are the only possibilities for fusion systems. Each of the three cases satisfies the hypotheses of Lemma 5.8 and so represent saturated fusion systems. Realizing the third cases as a fusion system of a finite simple group follows from [10, Proposition 1.64]. In the second case, the unique involution in $Z(S)$ is not $\mathcal{F}$-conjugate to any other element of S; therefore, $\mathcal{F}$ has a non-trivial center. The first case is easily seen to be the fusion system of $V : \Sigma_3$ since $V \lhd S$ and V is the unique proper $\mathcal{F}$-Alperin subgroup. $\square$

We finish the paper by discussing saturated fusion systems over semi-dihedral 2-groups. We start by classifying the possible proper $\mathcal{F}$-Alperin subgroups for a fusion system $\mathcal{F}$ over a semidihedral 2-group

Lemma 5.15. *Let S be a semi-dihedral 2-group and $\mathcal{F}$ a fusion system over S. If $P \subsetneq S$ is an $\mathcal{F}$-Alperin subgroup, then P is either an elementary abelian subgroup of rank two or a quaternion eight group.*

Proof. The rank two elementary abelian subgroups of S and the quaternion eight subgroups of S are the only 2-Alperin subgroups of S. $\square$

We end by classifying saturated fusion systems over semi-dihedral 2-groups. Theorem 5.16 completes the proof of Theorem 1.1. Before we give Theorem 5.16, we need to define the groups $PGL_2^*(\mathbb{F}_{q^2})$ as in [12]. Let $G = P\Gamma L_2(\mathbb{F}_{q^2})$ be the automorpism group of $H = PSL_2(\mathbb{F}_{q^2})$. Let $L = PGL_2(\mathbb{F}_{q^2})$. G is of the form LF, where F is cyclic of order 2, induced from the semilinear transformations of the natural vector space on which $GL_2(\mathbb{F}_{q^2})$ acts. G has three subgroups each containing H as an index 2. The first two are L and HF. The third group is what is denoted as $PGL_2^*(\mathbb{F}_{q^2})$ and has a semidihedral Sylow 2-subgroup.

Theorem 5.16. *Let $S = SD_{2^n}$ be the semi-dihedral group of order 2^n, $n \geq 4$. S has one conjugacy class of rank two elementary abelian subgroups which we will represent by E and one conjugacy class of quaternion eight subgroups which we will represent by Q. If $\mathcal{F}$ is a nontrivial saturated fusion system over S, then then $\mathrm{Out}_{\mathcal{F}}(S) = 1$ and either*

1) the rank two elementary abelian subgroups of S are the only proper $\mathcal{F}$-Alperin subgroups of S, $\mathrm{Out}_{\mathcal{F}}(E) \cong \Sigma_3$, and $\mathrm{Out}_{\mathcal{F}}(Q) \cong \mathbb{Z}_2$;

2) the quaternion eight subgroups of S are the only proper $\mathcal{F}$-Alperin subgroups of S, $\mathrm{Out}_{\mathcal{F}}(Q) \cong \Sigma_3$, and $\mathrm{Out}_{\mathcal{F}}(E) \cong \mathbb{Z}_2$; or

3) the quaternion eight subgroups together with the rank two elementary abelian subgroups of

S are the only proper $\mathcal{F}$-Alperin subgroups of S, $\mathrm{Out}_{\mathcal{F}}(Q) \cong \Sigma_3$*, and* $\mathrm{Out}_{\mathcal{F}}(E) \cong \Sigma_3$*.*
In the first case, $PGL_2^*(\mathbb{F}_{q^2})$ for some prime q with $q \equiv 2^{n-1} + 1(mod\ 2^n)$. In the second case, $\mathcal{F} = \mathcal{F}_S(GL_2(\mathbb{F}_q))$ with q a prime such that $q \equiv 2^{n-2} - 1(mod\ 2^{n-1})$. In the third case, $\mathcal{F} = \mathcal{F}_S(PSL_3(\mathbb{F}_q))$ with q a prime such that $q \equiv 2^{n-2} - 1(mod\ 2^{n-1})$.

Proof. Since S has no odd order automorphisms, $\mathrm{Out}_{\mathcal{F}}(S) = 1$. By Lemma 5.15, any $\mathcal{F}$-Alperin subgroup of S is either rank two elementary abelian or a quaternion eight group. By Proposition 5.11 and Lemma 5.13, the three cases in the theorem are the only possibilities for fusion systems. Each of the three cases satisfies the hypotheses of Lemma 5.8 and so represent saturated fusion systems. Realizing the third cases as a fusion system of a finite simple group follows from [10, Proposition 1.64]. Let $\mathcal{F}$ be the fusion system in the second case and notice that $Z(S) = Z_{\mathcal{F}}(S)$. We see that $S/Z(S)$ is a dihedral 2-group of order 2^{n-1} and that the fusion system $\mathcal{F}/Z(S)$ over $S/Z(S)$ is the second type in Thoerem 5.12. Since $\mathcal{F}/Z(S)$ is the fusion system of a finite group, $\mathcal{F}$ is also the fusion system of a finite group by Lemma 3.16. By examining the case two in Thoerem 5.12, it is clear that $\mathcal{F} = \mathcal{F}_S(GL_2(\mathbb{F}_q))$ with q a prime such that $q \equiv 2^{n-2} - 1(\mathrm{mod}\ 2^{n-1})$. Let $\mathcal{F}$ be as in the first case. From [3, Lemma 4], we see that $\mathcal{F} = \mathcal{F}_S(PGL_2^*(\mathbb{F}_{q^2}))$ with q a prime such that $q^2 \equiv 2^{n-1} + 1(\mathrm{mod}\ 2^n)$. □

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