We will use the notation and names laid out by Broto, Levi and Oliver in [2].

**Definition 1.**

1. A category $\mathcal{F}$ over a $p$-group $S$ is a category with objects the subgroups of $S$ and morphisms in $\mathcal{F}$ are group monomorphisms between the subgroups.
2. A category $\mathcal{F}$ over $S$ is called a restrictive category if for each $P' \leq P \leq S$ and $Q' \leq Q \leq S$, and each $\varphi \in \text{Hom}_\mathcal{F}(P, Q)$ such that $\varphi(P') \leq Q'$, $\varphi|_{P'} \in \text{Hom}_\mathcal{F}(P', Q')$.
3. For a restrictive category $\mathcal{F}$ over $S$ and any subgroup $A \leq \text{Aut}(S)$, $\langle \mathcal{F}, A \rangle$ is the smallest restrictive category over $S$ which contains $\mathcal{F}$ together with all automorphisms in $A$ and their restrictions.

Parts 2 and 3 of the definition are from Definition 3.2 of [1].

For notation we will use $(S, \mathcal{F})$ to mean a $\mathcal{F}$ is a (restrictive) category over $S$.

**Definition 2.** A saturated fusion system on a finite $p$-group $P$ is a restrictive category $\mathcal{F}$ on $P$ such that the following properties hold:

- **SFS1:** For any two subgroups $Q, R$ of $P$, $\text{Hom}_P(Q, R) \subseteq \text{Hom}_\mathcal{F}(Q, R)$.
- **SFS2:** If $Q \subseteq P$ and $\varphi \in \text{Hom}_\mathcal{F}(Q, P)$ are such that $\varphi(Q)$ is fully $\mathcal{F}$-normalized, $\varphi$ can be extended to $\bar{\varphi} : N_\varphi \to P$ where $N_\varphi = \{g \in N_P(Q) | \varphi c g \varphi^{-1} \in \text{Aut}_P(\varphi(Q)) \}$.
- **SFS3:** $\text{Aut}_P(P) \in \text{Syl}(\text{Aut}_\mathcal{F}(P))$.

**Definition 3.** Let $(S, \mathcal{F})$ and $(P, \mathcal{F}')$ be categories with $P \leq S$. 
(1) We say that $P$ is strongly $\mathcal{F}$-closed if for any morphism $\varphi : Q \to Q'$ in $\mathcal{F}$, $\varphi|_{P\cap Q}(P \cap Q) \leq P$.

(2) We say that $\mathcal{F}$ normalizes $\mathcal{F}'$ if $P$ is strongly $\mathcal{F}$-closed and for every isomorphism $\varphi : Q \to Q'$ in $\mathcal{F}$ and any two subgroups $R, R'$ of $Q \cap P$ we have

$$\varphi \circ \text{Hom}_{\mathcal{F}}(R, R') \circ \varphi^{-1}|_{\varphi(R)} \subseteq \text{Hom}_{\mathcal{F}'}(\varphi(R), \varphi(R')).$$

(3) We say $\mathcal{F}'$ is normal in $\mathcal{F}$ if $(P, \mathcal{F}') \subseteq (S, \mathcal{F})$ and $\mathcal{F}$ normalizes $\mathcal{F}'$. We denote this $\mathcal{F}' \leq \mathcal{F}$ or $(P, \mathcal{F}') \leq (S, \mathcal{F})$.

Parts two and three here are in section 3 of [3]. (Ref. for part 1)

**Definition 4.** Let $(S, \mathcal{F})$ be a restrictive category and $P \leq S$ be strongly $\mathcal{F}$-closed. We use $\mathcal{F}|_P$ to denote the restrictive category on $P$ that is the full subcategory of $\mathcal{F}$ whose objects are subgroups of $P$.

**Remark 5.** If $(P, \mathcal{F}') \leq (S, \mathcal{F})$ are both categories then for $Q \leq P$, $\text{Aut}_{\mathcal{F}'}(Q) \leq \text{Aut}_{\mathcal{F}}(Q)$ and so $(P, \mathcal{F}') \leq (P, \mathcal{F}|_P)$

**Remark 6.** If $(P, \mathcal{F}')$ and $(H, \mathcal{D})$ are two restrictive categories that are normal in $(S, \mathcal{F})$, then $(P, \mathcal{F}') \cap (H, \mathcal{D})$, which is the restrictive category over $P \cap H$ whose morphisms are all of the morphisms contained in both $\mathcal{F}'$ and $\mathcal{D}$, is normal in $(S, \mathcal{F})$.

**Definition 7** ([3, Definition 4.1]). A saturated fusion system $(S, \mathcal{F})$ is called *simple* if it has no proper non trivial normal saturated fusion systems.

**Definition 8.** A saturated fusion system $(S, \mathcal{F})$ is called *elementary* if $S$ has no proper nontrivial strongly $\mathcal{F}$-closed subgroups.

**Remark 9.** Let $(S, \mathcal{F})$ be a saturated fusion system. A nontrivial proper subgroup $P \subset S$ is strongly $\mathcal{F}$-closed if and only if the restrictive category $(P, \mathcal{F}|_P)$ is normalized by $\mathcal{F}$.

**Definition 10.** For a category $(S, \mathcal{F})$ let $\text{Aut}_{\text{fus}}(S, \mathcal{F})$ be the group of automorphisms in $\text{Aut}(S)$ which normalize $(S, \mathcal{F})$. (As for any subgroup $A \leq \text{Aut}(S)$, $A$ is a category over $S$.)
This definition is used in [1], although formulated slightly differently.

**Definition 11.** Let \((S, \mathcal{F})\) be a restrictive category. We say that a restrictive category \((P, \mathcal{F}') \subseteq (S, \mathcal{F})\) is characteristic if \((P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})\) and \(\text{Aut}_{\text{fus}}(S, \mathcal{F})\) normalizes \((P, \mathcal{F}')\). We denote this \((P, \mathcal{F}')\) char \((S, \mathcal{F})\).

**Definition 12.** Let \((S, \mathcal{F})\) be a saturated fusion system. We say that a proper subgroup \(P \subseteq S\) is ingrained if \(P\) is strongly \(\mathcal{F}\)-closed and \(P\) is fixed by \(\text{Aut}_{\text{fus}}(S, \mathcal{F})\). We would say then that \(P\) is \(\mathcal{F}\)-ingrained.

**Remark 13.** Let \((S, \mathcal{F})\) be a saturated fusion system. A nontrivial proper subgroup \(P \subset S\) is \(\mathcal{F}\)-ingrained if and only if the restrictive category \((P, \mathcal{F}|_P)\) is characteristic in \((S, \mathcal{F})\).

**Theorem 14.** Let \((P, \mathcal{F}')\) and \((S, \mathcal{F})\) be saturated fusion systems. If \((P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})\), then \(\mathcal{F}|_P = \langle \mathcal{F}', \text{Aut}_{\mathcal{F}}(P) \rangle\).

The proof of Theorem 14 is similar to the proof of Lemma 3.4 (b) of [1].

**Proof.** \(\mathcal{F}|_P = \langle \mathcal{F}', \text{Aut}_{\mathcal{F}}(P) \rangle\) is equivalent to every morphisms \(\mathcal{F}|_P\) being a composition of morphisms in \(\mathcal{F}'\) and restrictions of elements in \(\text{Aut}_{\mathcal{F}}(P)\). By Alperin’s fusion theorem for saturated fusion systems it is enough to show that for every \(Q \leq S\) and every \(\varphi \in \text{Aut}_{\mathcal{F}}(Q)\), \(\varphi|_{P \cap Q}\) is a composition of morphisms in \(\mathcal{F}'\) and restrictions of elements in \(\text{Aut}_{\mathcal{F}}(P)\). Since \(P\) is strongly \(\mathcal{F}\)-closed \(\varphi|_{P \cap Q} \in \text{Aut}_{\mathcal{F}}(P \cap Q)\); therefore, it is enough to show that for any subgroup \(Q \leq P\), every element of \(\text{Aut}_{\mathcal{F}}(Q)\) is a composition of morphisms in \(\mathcal{F}'\) and restrictions of elements in \(\text{Aut}_{\mathcal{F}}(P)\).

Clearly \(\text{Aut}_{\mathcal{F}}(P) \subseteq \langle \mathcal{F}', \text{Aut}_{\mathcal{F}}(P) \rangle\). Let \(Q \subseteq P\) and assume inductively that for all \(Q' \leq P\) with \(|Q'| > |Q|\), \(\text{Aut}_{\mathcal{F}}(Q') \subseteq \langle \mathcal{F}', \text{Aut}_{\mathcal{F}}(P) \rangle\). Fix \(Q\) and \(\varphi \in \text{Aut}_{\mathcal{F}}(Q)\). \(Q\) must be \(\mathcal{F}'\)-conjugate to a fully normalized (in \(\mathcal{F}'\)) subgroup \(H\) of \(P\). Say \(\psi : Q \rightarrow H\) is an isomorphism in \(\mathcal{F}'\) and let \(\varphi' = \psi \circ \varphi \circ \psi^{-1} \in \text{Aut}_{\mathcal{F}}(H)\). Since \(H\) is fully normalized (in \(\mathcal{F}'\)), \(\text{Aut}_{\mathcal{F}}(H) \in \text{Syl}_p(\text{Aut}_{\mathcal{F}}(H))\).
Consider $K = \varphi' \text{Aut}_P(H)(\varphi')^{-1} = \{\varphi' c_g(\varphi')^{-1} \in \text{Aut}_F(H) \mid g \in N_P(H)\}$. Clearly $|K| = |\text{Aut}_P(H)|$; thus, $K$ is also in $\text{Syl}_p(\text{Aut}_F(H))$. So there exists $\chi \in \text{Aut}_F(H)$ such that $\chi K \chi^{-1} = \text{Aut}_P(H)$.

So $N_{\chi \varphi'} \supseteq \{g \in N_P(H) \mid (\chi \varphi') c_g (\chi \varphi')^{-1} \in \text{Aut}_P(H)\} = N_P(H)$. By the extension axiom for saturated fusion systems, $\chi \circ \varphi'$ can be extended to a homomorphism $\lambda \in \text{Hom}_F(N_P(H), P)$. By the induction hypothesis (since $\lambda$ will be the restriction of an $F$-automorphism of some subgroup of $P$ containing $N_P(H)$), $\lambda \in \langle F', \text{Aut}_F(P) \rangle$. So also $\chi \circ \varphi' = \chi \circ \psi^{-1} \circ \varphi \circ \psi \in \langle F', \text{Aut}_F(P) \rangle$. Since $\chi$ and $\psi$ are both morphisms in $F'$, $\varphi \in \langle F', \text{Aut}_F(P) \rangle$. □

**Lemma 15.** Let $(P, F')$ and $(S, F)$ be saturated fusion systems with $(P, F') \subseteq (S, F)$. Suppose that $K$ is a proper subgroup of $P$, then $K$ is strongly $F$-closed if and only if $K$ is strongly $F'|_P$-closed. In addition suppose that $(H, D)$ is a restrictive category with $H \subseteq P$, then $D$ is normalized by $F$ if and only if $D$ is normalized by $F'|_P$.

**Proof.** Clearly if $K$ is strongly $F$-closed, then $K$ is strongly $F'|_P$-closed. Since any morphism in $F$ must take any element of $K$ to an element in $P$ (since $P$ is strongly $F$-closed), if $K$ is strongly $F'|_P$-closed it must also be strongly $F$-closed.

Clearly if $F$ normalizes $D$ then so does $F|_P$. Assume that $F|_P$ normalizes $D$. Let $\varphi : Q \to Q'$ be a morphism in $F$. $\varphi|_{H \cap Q}(H \cap Q) = (\varphi|_P)|_{H \cap Q}(H \cap Q) \leq H$. If $R, R' \leq H \cap Q$, $\varphi \circ \text{Hom}_D(R, R') \circ \varphi^{-1}|_{\varphi(R)} = \varphi|_P \circ \text{Hom}_F(R, R') \circ (\varphi^{-1}|_P)|_{\varphi(R)} \subseteq \text{Hom}_F(\varphi(R), \varphi(R'))$. □

**Lemma 16.** Let $(P, F')$ and $(S, F)$ be saturated fusion systems and let $(H, D)$ a restrictive category. If $(H, D)$ char $(P, F') \subseteq (S, F)$, then $(H, D) \subseteq (S, F)$.

**Proof.** By Lemma 15, it is enough to show that $D$ is normalized by $F|_P$. By Theorem 14, we have that $F|_P = \langle F', \text{Aut}_F(P) \rangle$. Since $(H, D)$ char $(P, F')$, $D$ is normalized by $F'$ and by $\text{Aut}_{\text{fus}}(P, F')$. Clearly $\text{Aut}_F(P) \subseteq \text{Aut}_{\text{fus}}(P, F')$ and so the result follows. □

**Proposition 17.** Let $(P, F')$ and $(S, F)$ be saturated fusion systems with $(P, F') \subseteq (S, F)$. Let $H \subset P$ be a proper nontrivial subset. If $H$ is $F'$-ingrained, then $H$ is strongly $F$-closed.
Proof. By Lemma 15, it is enough to show that $H$ strongly $\mathcal{F}|_P$-closed. By Theorem 14, it is enough to show that $H$ is both strongly $\mathcal{F}'$-closed and is fixed by Aut$_\mathcal{F}(P)$. Since $H$ is $\mathcal{F}'$-ingrained, $H$ is strongly $\mathcal{F}'$-closed by definition. In addition since $\mathcal{F}'$ is normalized by $\mathcal{F}$, Aut$_\mathcal{F}(P) \subset$ Aut$_{\text{fus}}(P, \mathcal{F}')$. Thus again by definition $H$ is fixed by Aut$_\mathcal{F}(P)$. □

Remark 18. It is also easy to show that the property of being characteristic is transitive for saturated fusion systems. (Write this out.)

Remark 19. For two restrictive categories $(H, \mathcal{D})$ and $(P, \mathcal{F})$, $(H, \mathcal{D}) \times (P, \mathcal{F})$ is the direct product and is again a restrictive category over $H \times P$. The direct product of saturated fusion systems is again a saturated fusion system.

Lemma 20. Let $(S, \mathcal{F})$ be a restrictive category. If $P$ and $Q$ are strongly $\mathcal{F}$-closed subgroups of $S$, then $P \cap Q$ is strongly $\mathcal{F}$-closed.

Lemma 21. Let $(S, \mathcal{F})$ be a saturated fusion system, such that $S = S_1 \times S_2 \times \cdots \times S_n$ with each $S_i$ strongly $\mathcal{F}$-closed, then for each $i = 1, 2, \ldots, n$ the restrictive category $(S_i, \mathcal{F}|_{S_i})$ is a saturated fusion system and is normal in $(S, \mathcal{F})$.

Proof. For a given $i$, let the restrictive category $(S_i, \mathcal{F}|_{S_i})$ be $(P, \mathcal{F}')$. Let $\tilde{S} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$. (Using the formulation of saturated fusion systems given by Linckelmann.) Let $A = \{ \varphi \in \text{Aut}_\mathcal{F}(S) | \varphi|_S = \text{id}_S \}$. Clearly since $P$ is strongly $\mathcal{F}$-closed and $\tilde{S} \subseteq C_S(P)$, Aut$_{\mathcal{F}'}(P) \cong A$. Since $\tilde{S}$ is strongly $\mathcal{F}$-closed it follows that $A \subseteq$ Aut$_\mathcal{F}(S)$. Also $A \cap$ Aut$_S(S) \cong$ Aut$_P(P)$ which then implies that Aut$_P(P)$ is a Sylow $p$-subgroup of Aut$_{\mathcal{F}'}(P)$ since Aut$_S(S)$ is a Sylow $p$-subgroup of Aut$_\mathcal{F}(S)$. Now let $\varphi : Q \rightarrow P$ be a morphism in $\mathcal{F}'$ such that $\varphi(Q)$ is fully $\mathcal{F}'$-normalized. Since $P$ is strongly $\mathcal{F}$-closed and $\tilde{S} \subseteq C_S(R)$ for any subgroup $R \subseteq P$, we see that $\varphi(Q)$ is fully $\mathcal{F}$-normalized and $\varphi$ is a morphism in $\mathcal{F}$. Since $(S, \mathcal{F})$ is a saturated fusion system $\varphi$ extends to a morphism $\psi : N_\varphi^S \rightarrow S$ in $\mathcal{F}$. Then $\psi|_P : N_\varphi^S \cap P \rightarrow P$ is a morphism in $\mathcal{F}'$ also extending $\varphi$. It is enough no to realize that $N_\varphi^S \cap P = N_\varphi^P$. □
Theorem 22. Let $(S, \mathcal{F})$ be a saturated fusion system. If $S$ has no proper nontrivial $\mathcal{F}$-ingrained subgroup, then $(S, \mathcal{F})$ is either an elementary saturated fusion system or $S$ is the direct product of some number of isomorphic subgroups $S_i$ such that the restrictive categories $(S_i, \mathcal{F}|_{S_i})$ are isomorphic elementary saturated fusion systems.

Proof. If $S$ does not have a proper non-trivial strongly $\mathcal{F}$-closed subgroup, $(S, \mathcal{F})$ is an elementary saturated fusion system by definition. Suppose on the other hand that $S$ has a proper non-trivial strongly $\mathcal{F}$-closed subgroup. Choose a minimal nontrivial strongly $\mathcal{F}$-closed subgroup $H \subset S$. Consider all of the subgroups of $S$ of the form $\prod_{i=1}^{n} H_i$ where each $H_i \cong H$ and $H_i$ is strongly $\mathcal{F}$-closed in $S$. Let $M$ be such a subgroup of the largest possible $n$. Clearly $M$ is strongly $\mathcal{F}$-closed. Let $\varphi \in \text{Aut}_{\text{fus}}(S, \mathcal{F})$ and $1 \leq i \leq n$. $\varphi(H_i) \cong (H_i) \cong (H)$ and $\varphi(H_i)$ is strongly $\mathcal{F}$-closed.

Suppose that $\varphi(H_i) \not\subseteq M$. Then $\varphi(H_i) \cap M$ is a strongly $\mathcal{F}$-closed subgroup of $S$ which is smaller than $H$ and so the intersection must be trivial. Thus $\varphi(H_i) \times M \subseteq S$ is of the same type as $M$ but with $n$ one larger. implies that for any $\varphi$ This contradicts the choice of $n$. So $\varphi(H_i) \subseteq M$. And so $\varphi$ normalizes $M$; therefore, $M$ is $\mathcal{F}$-ingrained which implies that $M = S$.

Notice that in the above argument there is contained the argument that if $\varphi \in \text{Aut}_{\text{fus}}(S, \mathcal{F})$ then for any $i$, $\varphi(H_i) = (H_j)$ for some $j$.

By Lemma 21, we see then that for each $i$, $(H_i, \mathcal{F}|_{H_i})$ is a saturated fusion subsystem normal in $(S, \mathcal{F})$. Suppose that for some $i \neq j$, $(H_i, \mathcal{F}|_{H_i})$ and $(H_j, \mathcal{F}|_{H_j})$ are not isomorphic saturated fusion systems. This implies that there cannot exist a $\varphi \in \text{Aut}_{\text{fus}}(S, \mathcal{F})$ with $\varphi(H_i) = \varphi(H_j)$. So let $Q$ be the subset of $S$ generated by all images under elements $\varphi \in \text{Aut}_{\text{fus}}(S, \mathcal{F})$ of $H_i$. Clearly $Q$ is a proper nontrivial subset of $S$ and $Q$ is $\mathcal{F}$-ingrained. This contradicts the hypotheses of the theorem, so for each $i \neq j$ $(H_i, \mathcal{F}|_{H_i})$ and $(H_j, \mathcal{F}|_{H_j})$ must be isomorphic saturated fusion systems. □
Corollary 23. Let \((S, \mathcal{F})\) be a saturated fusion system. If \(P\) is a minimal proper non-trivial strongly \(\mathcal{F}\)-closed subgroup of \(S\) and \((P, \mathcal{F}')\) is any saturated fusion system over \(P\) that is normalized by \(\mathcal{F}\), then \((P, \mathcal{F}')\) is either elementary or \(P\) is the direct product of some number of isomorphic groups \(S_i\) such that the restrictive \((S_i, \mathcal{F}'|_{S_i})\) are isomorphic elementary saturated fusion systems.

Proof. By Theorem 22, it is enough to show that \(P\) does not contain a proper non-trivial \(\mathcal{F}'\)-ingrained subgroup. Suppose that \(P\) has a proper non-trivial \(\mathcal{F}'\)-ingrained subgroup \(H\). Then since \((P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})\), \(H\) must be a non-trivial strongly \(\mathcal{F}\)-closed subgroup of \(S\), which contradicts the minimality of \(P\).

An important theorem of Linckelmann, shows that if a simple fusion system is the saturated fusion system of some finite group, then it is the fusion system of a finite simple group.

Theorem 24 ([3, Proposition 4.2]). Let \((P, \mathcal{F})\) be a simple saturated fusion. Suppose that \(\mathcal{F} = \mathcal{F}_P(G)\) for some finite group \(G\) with \(P \in \text{Syl}_p(G)\). If \(O_{p'}(G) = 1\) and if \(\mathcal{F}_P(G) \neq \mathcal{F}_P(H)\) for any proper subgroup \(H\) of \(G\) with \(P \in \text{Syl}_p(H)\), then \(G\) is simple. In particular, if \(G\) has minimal order such that \(P \in \text{Syl}_p(G)\) and such that \(\mathcal{F} = \mathcal{F}_P(G)\), then \(G\) is simple.

Theorem 25. Let \((P, \mathcal{F})\) be an elementary saturated fusion system such that \((P, \mathcal{F})\) has no proper normal fusion system of the form \((P, \mathcal{F}')\). Suppose \(\mathcal{F} = \mathcal{F}_P(G)\) for some finite group \(G\) with \(P \in \text{Syl}_p(G)\). If \(O_{p'}(G) = 1\) and if \(\mathcal{F}_P(G) \neq \mathcal{F}_P(H)\) for any proper subgroup \(H\) of \(G\) with \(P \in \text{Syl}_p(H)\), then \(G\) is simple. In particular, if \(G\) has minimal order such that \(P \in \text{Syl}_p(G)\) and such that \(\mathcal{F} = \mathcal{F}_P(G)\), then \(G\) is simple.

Proof. Suppose \(O_{p'}(G) = 1\) and \(\mathcal{F}_P(G) \neq \mathcal{F}_P(H)\) for any proper subgroup \(H\) of \(G\) with \(P \in \text{Syl}_p(H)\). Suppose that \(G\) is not simple. Let \(N\) be the minimal non-trivial normal subgroup of \(G\). If \(Q = N \cap P\), then \(Q \in \text{Syl}_p(N)\) and \(Q\) is strongly \(\mathcal{F}\)-closed. Since \(O_{p'}(G) = 1\), \(N\) is not a \(p'\)-group; thus, \(Q = P\) since \((P, \mathcal{F})\) is elementary. We have \((P, \mathcal{F}_P(N)) \trianglelefteq (P, \mathcal{F}_P(G)) = (P, \mathcal{F})\). However this is not allowed unless the fusion systems are the same, and so \((P, \mathcal{F}_P(N)) = (P, \mathcal{F})\). Therefore, by hypothesis \(N = G\), showing that \(G\) is
Let $G$ be a finite group of minimal order such that $P \in \text{Syl}_p(G)$ and such that $\mathcal{F} = \mathcal{F}_P(G)$. Then $O_{p'}(G) = 1$, because the canonical map $G \to G/O_{p'}(G)$ induces an isomorphism of saturated fusion systems. By the minimality of $G$, we have that $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup $H$ of $G$ with $P \in \text{Syl}_p(H)$. The second statement, thus, follows from the first.

Also we can use a similar Theorem to Theorem 9.2 of [2].

**Theorem 26** (See [2, Theorem 9.2]). Let $(P, \mathcal{F})$ be an elementary saturated fusion system over a non-abelian $p$-group $S$. Suppose $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group $G$ with $P \in \text{Syl}_p(G)$. If $O_{p'}(G) = 1$ and if $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup $H$ of $G$ with $P \in \text{Syl}_p(H)$, then $G$ is a finite almost simple group or $G$ has a proper normal subgroup $H$ containing $P$ (as a Sylow subgroup) that is the product of isomorphic finite simple groups and $H \trianglelefteq G \subseteq \text{Aut}(H)$.

**Proof.** Suppose $O_{p'}(G) = 1$ and $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup $H$ of $G$ with $P \in \text{Syl}_p(H)$. Let $N$ be the minimal non-trivial normal subgroup of $G$. If $Q = N \cap P$, then $Q \in \text{Syl}_p(N)$ and $Q$ is strongly $\mathcal{F}$-closed. Since $O_{p'}(G) = 1$, $N$ is not a $p'$-group; thus, $Q = P$ since $(P, \mathcal{F})$ is elementary.

By the minimality of $N$ above normal subgroups, $N$ is the product of isomorphic non-abelian simple groups which are permuted transitively by $N_G(N) = G$ (otherwise $N$ is not minimal). Thus $C_G(N) \cap N = 1$, and so $C_G(N) = 1$ by the assumption that $O_{p'}(G) = 1$. This shows that $H \trianglelefteq G \subseteq \text{Aut}(H)$. If $H$ itself is simple, then $G$ is almost simple. If on the other hand, $H$ is the product of simple groups and is normal in $G$.

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Theorem 27. Let \((P, \mathcal{F})\) be a saturated fusion system such that \(P\) has no proper nontrivial \(\mathcal{F}\)-ingrained subgroup. Suppose \(\mathcal{F} = \mathcal{F}_P(G)\) for some finite group \(G\) with \(P \in \text{Syl}_p(G)\). If \(O_{p'}(G) = 1\) and if \(\mathcal{F}_P(G) \neq \mathcal{F}_P(H)\) for any proper subgroup \(H\) of \(G\) with \(P \in \text{Syl}_p(H)\), then \(G\) is a direct product of simple groups. In particular, if \(G\) has minimal order such that \(P \in \text{Syl}_p(G)\) and such that \(\mathcal{F} = \mathcal{F}_P(G)\), then \(G\) is a direct product of simple groups.

Proof. By Theorem 22, \(P = S_1 \times S_2 \times \cdots \times S_n\), where each \((S_i, \mathcal{F}|_{S_i})\) are isomorphic and elementary saturated fusion systems.

Suppose \(O_{p'}(G) = 1\) and \(\mathcal{F}_P(G) \neq \mathcal{F}_P(H)\) for any proper subgroup \(H\) of \(G\) with \(P \in \text{Syl}_p(H)\).

For each \(i\), let \(N_i = \langle S_i^G \rangle\). Each \(N_i\) is normal in \(G\) by definition and is a minimal normal subgroup. If \(\emptyset \neq H \subseteq N_i\) is normal in \(H\), then \(H \cap S_i\) is strongly \(\mathcal{F}\)-closed and is not empty since \(O_{p'}(G) = 1\). Thus \(H = N_i\).

So each \(N_i\) is characteristically simple. If \(S_j \cap N_i \neq \emptyset\), then \(N_j = N_i\).

\[\square\]

References


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