ELEMENTARY SATURATED FUSION SUBSYSTEMS AND INGRAINED SUBGROUPS

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More Intro

We will use the notation and names laid out by Broto, Levi and Oliver in [2].

Definition 1. (1) A category $\mathcal{F}$ over a p-group S is a category with objects the subgroups of S and morphisms in $\mathcal{F}$ are group monomorphisms between the subgroups.

(2) A category $\mathcal{F}$ over S is called a restrictive category if for each $P' \leq P \leq S$ and $Q' \leq Q \leq S$, and each $\varphi \in \mathrm{Hom}_{\mathcal{F}}(P, Q)$ such that $\varphi(P') \leq Q'$, $\varphi|_{P'} \in \mathrm{Hom}_{\mathcal{F}}(P', Q')$.

(3) For a restrictive category $\mathcal{F}$ over S and any subgroup $A \leq \mathrm{Aut}(S)$, $\langle \mathcal{F}, A \rangle$ is the smallest restrictive category over S which contains $\mathcal{F}$ together with all automorphisms in A and their restrictions.

Parts 2 and 3 of the definition are from Definition 3.2 of [1].

For notation we will use $(S, \mathcal{F})$ to mean a $\mathcal{F}$ is a (restrictive) category over S.

Definition 2. A *saturated fusion system on a finite p-group* P is a restrictive category $\mathcal{F}$ on P such that the following properties hold:

SFS1: For any two subgroups Q, R of P, $\mathrm{Hom}_P(Q, R) \subseteq \mathrm{Hom}_{\mathcal{F}}(Q, R)$.

SFS2: If $Q \subseteq P$ and $\varphi \in \mathrm{Hom}_{\mathcal{F}}(Q, P)$ are such that $\varphi(Q)$ is fully $\mathcal{F}$-normalized, φ can be extended to $\bar{\varphi} : N_\varphi \to P$ where $N_\varphi = \{g \in N_P(Q) | \varphi c_g \varphi^{-1} \in \mathrm{Aut}_P(\varphi(Q))\}$.

SFS3: $\mathrm{Aut}_P(P) \in \mathrm{Syl}(\mathrm{Aut}_{\mathcal{F}}(P))$.

Definition 3. Let $(S, \mathcal{F})$ and $(P, \mathcal{F}')$ be categories with $P \leq S$.

(1) We say that P is strongly $\mathcal{F}$-closed if for any morphism $\varphi : Q \to Q'$ in $\mathcal{F}$, $\varphi|_{P\cap Q}(P \cap Q) \leq P$.

(2) We say that $\mathcal{F}$ normalizes $\mathcal{F}'$ if P is strongly $\mathcal{F}$-closed and for every isomorphism $\varphi : Q \to Q'$ in $\mathcal{F}$ and any two subgroups R, R' of $Q \cap P$ we have

$$\varphi \circ \mathrm{Hom}_{\mathcal{F}'}(R, R') \circ \varphi^{-1}|_{\varphi(R)} \subseteq \mathrm{Hom}_{\mathcal{F}'}(\varphi(R), \varphi(R')).$$

(3) We say $\mathcal{F}'$ is normal in $\mathcal{F}$ if $(P, \mathcal{F}') \subseteq (S, \mathcal{F})$ and $\mathcal{F}$ normalizes $\mathcal{F}'$. We denote this $\mathcal{F}' \trianglelefteq \mathcal{F}$ or $(P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})$.

Parts two and three here are in section 3 of [3]. (Ref. for part 1)

Definition 4. Let $(S, \mathcal{F})$ be a restrictive category and $P \leq S$ be strongly $\mathcal{F}$-closed. We use $\mathcal{F}|_P$ to denote the restrictive category on P that is the full subcategory of $\mathcal{F}$ whose objects are subgroups of P.

Remark 5. If $(P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})$ are both categories then for $Q \leq P$, $\mathrm{Aut}_{\mathcal{F}'}(Q) \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(Q)$ and so $(P, \mathcal{F}') \trianglelefteq (P, \mathcal{F}|_P)$

Remark 6. If $(P, \mathcal{F}')$ and $(H, \mathcal{D})$ are two restrictive categories that are normal in $(S, \mathcal{F})$, then $(P, \mathcal{F}') \cap (H, \mathcal{D})$, which is the restrictive category over $P \cap H$ whose morphisms are all of the morphisms contained in both $\mathcal{F}'$ and $\mathcal{D}$, is normal in $(S, \mathcal{F})$.

Definition 7 ([3, Definition 4.1])**.** A saturated fusion system $(S, \mathcal{F})$ is called *simple* if it has no proper non trivial normal saturated fusion systems.

Definition 8. A saturated fusion system $(S, \mathcal{F})$ is called *elementary* if S has no proper nontrivial strongly $\mathcal{F}$-closed subgroups.

Remark 9. Let $(S, \mathcal{F})$ be a saturated fusion system. A nontrivial proper subgroup $P \subset S$ is strongly $\mathcal{F}$-closed if and only if the restrictive category $(P, \mathcal{F}|_P)$ is normalized by $\mathcal{F}$.

Definition 10. For a category $(S, \mathcal{F})$ let $\mathrm{Aut}_{\mathrm{fus}}(S, \mathcal{F})$ be the group of automorphisms in $\mathrm{Aut}(S)$ which normalize $(S, \mathcal{F})$. (As for any subgroup $A \leq \mathrm{Aut}(S)$, A is a category over S.)

This definition is used in [1], although formulated slightly differently.

Definition 11. Let $(S, \mathcal{F})$ be a restrictive category. We say that a restrictive category $(P, \mathcal{F}') \subseteq (S, \mathcal{F})$ is *characteristic* if $(P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})$ and $\mathrm{Aut}_{\mathrm{fus}}(S, \mathcal{F})$ normalizes $(P, \mathcal{F}')$. We denote this $(P, \mathcal{F}')$ char $(S, \mathcal{F})$.

Definition 12. Let $(S, \mathcal{F})$ be a saturated fusion system. We say that a proper subgroup $P \subseteq S$ is *ingrained* if P is strongly $\mathcal{F}$-closed and P is fixed by $\mathrm{Aut}_{\mathrm{fus}}(S, \mathcal{F})$. We would say then that P is $\mathcal{F}$-ingrained.

Remark 13. Let $(S, \mathcal{F})$ be a saturated fusion system. A nontrivial proper subgroup $P \subset S$ is $\mathcal{F}$-ingrained if and only if the restrictive category $(P, \mathcal{F}|_P)$ is characteristic in $(S, \mathcal{F})$.

Theorem 14. *Let $(P, \mathcal{F}')$ and $(S, \mathcal{F})$ be saturated fusion systems. If $(P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})$, then $\mathcal{F}|_P = \langle \mathcal{F}', \mathrm{Aut}_{\mathcal{F}}(P) \rangle$.*

The proof of Theorem 14 is similar to the proof of Lemma 3.4 (b) of [1].

Proof. $\mathcal{F}|_P = \langle \mathcal{F}', \mathrm{Aut}_{\mathcal{F}}(P) \rangle$ is equivalent to every morphisms $\mathcal{F}|_P$ being a composition of morphisms in $\mathcal{F}'$ and restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(P)$. By Alperin's fusion theorem for saturated fusion systems it is enough to show that for every $Q \leq S$ and every $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$, $\varphi|_{P \cap Q}$ is a composition of morphisms in $\mathcal{F}'$ and restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(P)$. Since P is strongly $\mathcal{F}$-closed $\varphi|_{P \cap Q} \in \mathrm{Aut}_{\mathcal{F}}(P \cap Q)$; therefore, it is enough to show that for any subgroup $Q \leq P$, every element of $\mathrm{Aut}_{\mathcal{F}}(Q)$ is a composition of morphisms in $\mathcal{F}'$ and restrictions of elements in $\mathrm{Aut}_{\mathcal{F}}(P)$.

Clearly $\mathrm{Aut}_{\mathcal{F}}(P) \subseteq \langle \mathcal{F}', \mathrm{Aut}_{\mathcal{F}}(P) \rangle$. Let $Q \lneq P$ and assume inductively that for all $Q' \leq P$ with $|Q'| > |Q|$, $\mathrm{Aut}_{\mathcal{F}}(Q') \subseteq \langle \mathcal{F}', \mathrm{Aut}_{\mathcal{F}}(P) \rangle$. Fix Q and $\varphi \in \mathrm{Aut}_{\mathcal{F}}(Q)$. Q must be $\mathcal{F}'$-conjugate to a fully normalized (in $\mathcal{F}'$) subgroup H of P. Say $\psi : Q \to H$ is an isomorphism in $\mathcal{F}'$ and let $\varphi' = \psi \circ \varphi \circ \psi^{-1} \in \mathrm{Aut}_{\mathcal{F}}(H)$. Since H is fully normalized (in $\mathcal{F}'$), $\mathrm{Aut}_P(H) \in \mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}'}(H))$.

Consider $K = \varphi' \mathrm{Aut}_P(H)(\varphi')^{-1} = \{\varphi' c_g(\varphi')^{-1} \in \mathrm{Aut}_{\mathcal{F}'}(H) | g \in N_P(H)\}$. Clearly $|K| = |\mathrm{Aut}_P(H)|$; thus, K is also in $\mathrm{Syl}_p(\mathrm{Aut}_{\mathcal{F}'}(H))$. So there exists $\chi \in \mathrm{Aut}_{\mathcal{F}'}(H)$ such that $\chi K \chi^{-1} = \mathrm{Aut}_P(H)$.

So $N_{\chi\varphi'} \supseteq \{g \in N_P(H) | (\chi\varphi') c_g (\chi\varphi')^{-1} \in \mathrm{Aut}_P(H)\} = N_P(H)$. By the extension axiom for saturated fusion systems, $\chi \circ \varphi'$ can be extended to a homomorphism $\lambda \in \mathrm{Hom}_{\mathcal{F}}(N_P(H), P)$. By the induction hypothesis (since λ will be the restriction of an $\mathcal{F}$-automorphism of some subgroup of P containing $N_P(H)$), $\lambda \in \langle \mathcal{F}', \mathrm{Aut}_{\mathcal{F}}(P) \rangle$. So also $\chi \circ \varphi' = \chi \circ \psi^{-1} \circ \varphi \circ \psi \in \langle \mathcal{F}', \mathrm{Aut}_{\mathcal{F}}(P) \rangle$. Since χ and ψ are both morphisms in $\mathcal{F}'$, $\varphi \in \langle \mathcal{F}', \mathrm{Aut}_{\mathcal{F}}(P) \rangle$. □

Lemma 15. *Let $(P, \mathcal{F}')$ and $(S, \mathcal{F})$ be saturated fusion systems with $(P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})$. Suppose that K is a proper subgroup of P, then K is strongly $\mathcal{F}$-closed if and only if K is strongly $\mathcal{F}|_P$-closed. In addition suppose that $(H, \mathcal{D})$ is a restrictive category with $H \subseteq P$, then $\mathcal{D}$ is normalized by $\mathcal{F}$ if and only if $\mathcal{D}$ is normalized by $\mathcal{F}|_P$.*

Proof. Clearly if K is strongly $\mathcal{F}$-closed, then K is strongly $\mathcal{F}|_P$-closed. Since any morphism in $\mathcal{F}$ must take any element of K to an element in P (since P is strongly $\mathcal{F}$-closed), if K is strongly $\mathcal{F}|_P$-closed it must also be strongly $\mathcal{F}$- closed.

Clearly if $\mathcal{F}$ normalizes $\mathcal{D}$ then so does $\mathcal{F}|_P$. Assume that $\mathcal{F}|_P$ normalizes $\mathcal{D}$. Let $\varphi : Q \to Q'$ be a morphism in $\mathcal{F}$. $\varphi|_{H \cap Q}(H \cap Q) = (\varphi|_P)|_{H \cap Q}(H \cap Q) \leq H$. If $R, R' \leq H \cap Q$, $\varphi \circ \mathrm{Hom}_{\mathcal{F}'}(R, R') \circ \varphi^{-1}|_{\varphi(R)} = \varphi|_P \circ \mathrm{Hom}_{\mathcal{F}'}(R, R') \circ (\varphi^{-1}|_P)|_{\varphi(R)} \subseteq \mathrm{Hom}_{\mathcal{F}'}(\varphi(R), \varphi(R'))$. □

Lemma 16. *Let $(P, \mathcal{F}')$ and $(S, \mathcal{F})$ be saturated fusion systems and let $(H, \mathcal{D})$ a restrictive category. If $(H, \mathcal{D})$ char $(P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})$, then $(H, \mathcal{D}) \trianglelefteq (S, \mathcal{F})$.*

Proof. By Lemma 15, it is enough to show that $\mathcal{D}$ is normalized by $\mathcal{F}|_P$. By Theorem 14, we have that $\mathcal{F}|_P = \langle \mathcal{F}', \mathrm{Aut}_{\mathcal{F}}(P) \rangle$. Since $(H, \mathcal{D})$ char $(P, \mathcal{F}')$, $\mathcal{D}$ is normalized by $\mathcal{F}'$ and by $\mathrm{Aut}_{\mathrm{fus}}(P, \mathcal{F}')$. Clearly $\mathrm{Aut}_F(P) \subseteq \mathrm{Aut}_{\mathrm{fus}}(P, \mathcal{F}')$ and so the result follows. □

Proposition 17. *Let $(P, \mathcal{F}')$ and $(S, \mathcal{F})$ be saturated fusion systems with $(P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})$. Let $H \subset P$ be a proper nontrivial subset. If H is $\mathcal{F}'$-ingrained, then H is strongly $\mathcal{F}$-closed.*

Proof. By Lemma 15, it is enough to show that H strongly $\mathcal{F}|_P$-closed. By Theorem 14, it is enough to show that H is both strongly $\mathcal{F}'$-closed and is fixed by $\mathrm{Aut}_{\mathcal{F}}(P)$. Since H is $\mathcal{F}'$-ingrained, H is strongly $\mathcal{F}'$-closed by definition. In addition since $\mathcal{F}'$ is normalized by $\mathcal{F}$, $\mathrm{Aut}_{\mathcal{F}}(P) \subset \mathrm{Aut}_{\mathrm{fus}}(P, \mathcal{F}')$. Thus again by definition H is fixed by $\mathrm{Aut}_{\mathcal{F}}(P)$. □

Remark 18. It is also easy to show that the property of being characteristic is transitive for saturated fusion systems. (Write this out.)

Remark 19. For two restrictive categories $(H, \mathcal{D})$ and $(P, \mathcal{F})$, $(H, \mathcal{D}) \times (P, \mathcal{F})$ is the direct product and is again a restrictive category over $H \times P$. The direct product of saturated fusion systems is again a saturated fusion system.

Lemma 20. *Let $(S, \mathcal{F})$ be a restrictive category. If P and Q are strongly $\mathcal{F}$-closed subgroups of S, then $P \cap Q$ is strongly $\mathcal{F}$-closed.*

Lemma 21. *Let $(S, \mathcal{F})$ be a saturated fusion system, such that $S = S_1 \times S_2 \times \cdots \times S_n$ with each S_i strongly $\mathcal{F}$-closed, then for each $i = 1, 2, \ldots n$ the restrictive category $(S_i, \mathcal{F}|_{S_i})$ is a saturated fusion system and is normal in $(S, \mathcal{F})$.*

Proof. For a given i, let the restrictive category $(S_i, \mathcal{F}|_{S_i})$ be $(P, \mathcal{F}')$. Let $\bar{S} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots \times S_n$. (Using the formulation of saturated fusion systems given by Linckelmann.) Let $A = \{\varphi \in \mathrm{Aut}_{\mathcal{F}}(S) | \varphi|_{\bar{S}} = id_{\bar{S}}\}$. Clearly since P is strongly $\mathcal{F}$-closed and $\bar{S} \subseteq C_S(P)$, $\mathrm{Aut}_{\mathcal{F}'}(P) \cong A$. Since $\bar{S}$ is strongly $\mathcal{F}$-closed it follows that $A \trianglelefteq \mathrm{Aut}_{\mathcal{F}}(S)$. Also $A \cap \mathrm{Aut}_S(S) \cong \mathrm{Aut}_P(P)$ which then implies that $\mathrm{Aut}_P(P)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}'}(P)$ since $\mathrm{Aut}_S(S)$ is a Sylow p-subgroup of $\mathrm{Aut}_{\mathcal{F}}(S)$. Now let $\varphi : Q \to P$ be a morphism in $\mathcal{F}'$ such that $\varphi(Q)$ is fully $\mathcal{F}'$-normalized. Since P is strongly $\mathcal{F}$-closed and $\bar{S} \subseteq C_S(R)$ for any subgroup $R \subseteq P$, we see that $\varphi(Q)$ is fully $\mathcal{F}$-normalized and φ is a morphism in $\mathcal{F}$. Since $(S, \mathcal{F})$ is a saturated fusion system φ extends to a morphism $\psi : N_\varphi^S \to S$ in $\mathcal{F}$. Then $\psi|_P : N_\varphi^S \cap P \to P$ is a morphism in $\mathcal{F}'$ also extending φ. It is enough no to realize that $N_\varphi^S \cap P = N_\varphi^P$. □

Theorem 22. *Let $(S, \mathcal{F})$ be a saturated fusion system. If S has no proper nontrivial $\mathcal{F}$-ingrained subgroup, then $(S, \mathcal{F})$ is either an elementary saturated fusion system or S is the direct product of some number of isomorphic subgroups S_i such that the restrictive catagories $(S_i, \mathcal{F}|_{S_i})$ are isomorphic elementary saturated fusion systems.*

Proof. If S does not have a proper non-trivial strongly $\mathcal{F}$-closed subgroup, $(S, \mathcal{F})$ is an elementary saturated fusion system by definition. Suppose on the other hand that S has a proper non-trivial strongly $\mathcal{F}$-closed subgroup. Choose a minimal nontrivial strongly $\mathcal{F}$-closed subgroup $H \subset S$. Consider all of the subgroups of S of the form $\prod_{i=1}^{n} H_i$ where each $H_i \cong H$ and H_i is strongly $\mathcal{F}$-closed in S. Let M be such a subgroup of the largest possible n. Clearly M is strongly $\mathcal{F}$-closed. Let $\varphi \in \mathrm{Aut}_{\mathrm{fus}}(S, \mathcal{F})$ and $1 \leq i \leq n$. $\varphi(H_i) \cong (H_i) \cong (H)$ and $\varphi(H_i)$ is strongly $\mathcal{F}$-closed.

Suppose that $\varphi(H_i) \nsubseteq M$. Then $\varphi(H_i) \cap M$ is a strongly $\mathcal{F}$-closed subgroup of S which is smaller than H and so the intersection must be trivial. Thus $\varphi(H_i) \times M \subseteq S$ is of the some type as M but with n one larger. implies that for any φ This contradicts the choice of n. So $\varphi(H_i) \subseteq M$. And so φ normalizes M; therefore, M is $\mathcal{F}$-ingrained which implies that $M = S$.

Notice that in the above argument there is contained the argument that if $\varphi \in \mathrm{Aut}_{\mathrm{fus}}(S, \mathcal{F})$ then for any i, $\varphi(H_i) = (H_j)$ for some j.

By Lemma 21, we see then that for each i, $(H_i, \mathcal{F}|_{H_i})$ is a saturated fusion subsystem normal in $(S, \mathcal{F})$. Suppose that for some $i \neq j$, $(H_i, \mathcal{F}|_{H_i})$ and $(H_j, \mathcal{F}|_{H_j})$ are not isomorphic saturated fusion systems. This implies that there cannot exist a $\varphi \in \mathrm{Aut}_{\mathrm{fus}}(S, \mathcal{F})$ with $\varphi(H_i) = \varphi(H_j)$. So let Q be the subset of S generated by all images under elements $\varphi \in \mathrm{Aut}_{\mathrm{fus}}(S, \mathcal{F})$ of H_i. Clearly Q is a proper nontrivial subset of S and Q is $\mathcal{F}$-ingrained. This contradicts the hypotheses of the theorem, so for each $i \neq j$ $(H_i, \mathcal{F}|_{H_i})$ and $(H_j, \mathcal{F}|_{H_j})$ must be isomorphic saturated fusion systems. $\square$

Corollary 23. *Let $(S, \mathcal{F})$ be a saturated fusion system. If P is a minimal proper non-trivial strongly $\mathcal{F}$-closed subgroup of S and $(P, \mathcal{F}')$ is any saturated fusion system over P that is normalized by $\mathcal{F}$, then $(P, \mathcal{F}')$ is either elementary or P is the direct product of some number of isomorphic groups S_i such that the restrictive $(S_i, \mathcal{F}'|_{S_i})$ are isomorphic elementary saturated fusion systems.*

Proof. By Theorem 22, it is enough to show that P does not contain a proper non-trivial $\mathcal{F}'$-ingrained subgroup. Suppose that P has a proper non-trivial $\mathcal{F}'$-ingrained subgroup H. Then since $(P, \mathcal{F}') \trianglelefteq (S, \mathcal{F})$, H must be a non-trivial strongly $\mathcal{F}$-closed subgroup of S. which contradicts the minimality of P. □

An important theorem of Linckelmann, shows that if a simple fusion system is the saturated fusion system of some finite group, then it is the fusion system of a finite simple group.

Theorem 24 ([3, Proposition 4.2])**.** *Let $(P, \mathcal{F})$ be a simple saturated fusion. Suppose that $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G with $P \in \mathrm{Syl}_p(G)$. If $O_{p'}(G) = 1$ and if $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup H of G with $P \in \mathrm{Syl}_p(H)$, then G is simple. In particular, if G has minimal order such that $P \in \mathrm{Syl}_p(G)$ and such that $\mathcal{F} = \mathcal{F}_P(G)$, then G is simple.*

Theorem 25. *Let $(P, \mathcal{F})$ be an elementary saturated fusion system such that $(P, \mathcal{F})$ has no proper normal fusion system of the form $(P, \mathcal{F}')$. Suppose $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G with $P \in \mathrm{Syl}_p(G)$. If $O_{p'}(G) = 1$ and if $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup H of G with $P \in \mathrm{Syl}_p(H)$, then G is simple. In particular, if G has minimal order such that $P \in \mathrm{Syl}_p(G)$ and such that $\mathcal{F} = \mathcal{F}_P(G)$, then G is simple.*

Proof. Suppose $O_{p'}(G) = 1$ and $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup H of G with $P \in \mathrm{Syl}_p(H)$. Suppose that G is not simple. Let N be the minimal non-trivial normal subgroup of G. If $Q = N \cap P$, then $Q \in \mathrm{Syl}_p(N)$ and Q is strongly $\mathcal{F}$-closed. Since $O_{p'}(G) = 1$, N is not a p'-group; thus, $Q = P$ since $(P, \mathcal{F})$ is elementary. We have $(P, \mathcal{F}_P(N)) \lhd (P, \mathcal{F}_P(G)) = (P, \mathcal{F})$. However this is not allowed unless the fusion systems are the same, and so $(P, \mathcal{F}_P(N)) = (P, \mathcal{F})$. Therefore, by hypothesis $N = G$, showing that G is

simple.

Let G be a finite group of minimal order such that $P \in \mathrm{Syl}_p(G)$ and such that $\mathcal{F} = \mathcal{F}_P(G)$. Then $O_{p'}(G) = 1$, because the canonical map $G \to G/O_{p'}(G)$ induces an isomorphism of saturated fusion systems. By the minimality of G, we have that $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup H of G with $P \in \mathrm{Syl}_p(H)$. The second statement, thus, follows from the first. □

Also we can use a similar Theorem to Theorem 9.2 of [2].

Theorem 26 (See [2, Theorem 9.2])**.** *Let $(P, \mathcal{F})$ be an elementary saturated fusion system over a non-abelian p-group S. Suppose $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G with $P \in \mathrm{Syl}_p(G)$. If $O_{p'}(G) = 1$ and if $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup H of G with $P \in \mathrm{Syl}_p(H)$, then G is a finite almost simple group or G has a proper normal subgroup H containing P (as a Sylow subgroup) that is the product of isomorphic finite simple groups and $H \trianglelefteq G \subseteq \mathrm{Aut}(H)$.*

Proof. Suppose $O_{p'}(G) = 1$ and $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup H of G with $P \in \mathrm{Syl}_p(H)$. Let N be the minimal non-trivial normal subgroup of G. If $Q = N \cap P$, then $Q \in \mathrm{Syl}_p(N)$ and Q is strongly $\mathcal{F}$-closed. Since $O_{p'}(G) = 1$, N is not a p'-group; thus, $Q = P$ since $(P, \mathcal{F})$ is elementary.

By the minimality of N above normal subgroups, N is the product of isomorphic non-abelian simple groups which are permuted transitively by $N_G(N) = G$ (otherwise N is not minimal). Thus $C_G(N) \cap N = 1$, and so $C_G(N) = 1$ by the assumption that $O_{p'}(G) = 1$. This shows that $H \trianglelefteq G \subseteq \mathrm{Aut}(H)$. If H itself is simple, then G is almost simple. If on the other hand, H is the product of simple groups and is normal in G.

□

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Theorem 27. *Let $(P, \mathcal{F})$ be a saturated fusion system such that P has no proper nontrivial $\mathcal{F}$-ingrained subgroup. Suppose $\mathcal{F} = \mathcal{F}_P(G)$ for some finite group G with $P \in \mathrm{Syl}_p(G)$. If $O_{p'}(G) = 1$ and if $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup H of G with $P \in \mathrm{Syl}_p(H)$, then G is a direct product of simple groups. In particular, if G has minimal order such that $P \in \mathrm{Syl}_p(G)$ and such that $\mathcal{F} = \mathcal{F}_P(G)$, then G is a direct product of simple groups.*????????????

Proof. By Theorem 22, $P = S_1 \times S_2 \times \cdots \times S_n$, where each $(S_i, \mathcal{F}|_{S_i})$ are isomorphic and elementary saturated fusion systems.

Suppose $O_{p'}(G) = 1$ and $\mathcal{F}_P(G) \neq \mathcal{F}_P(H)$ for any proper subgroup H of G with $P \in \mathrm{Syl}_p(H)$. For each i, let $N_i = \langle S_i^G \rangle$. Each N_i is normal in G by definition and is a minimal normal subgroup. If $\emptyset \neq H \subseteq N_i$ is normal in H, then $H \cap S_i$ is strongly $\mathcal{F}$-closed and is not empty since $O_{p'}(G) = 1$. Thus $H = N_i$.

So each N_i is characteristically simple. If $S_j \cap N_i \neq \emptyset$, then $N_j = N_i$.

□

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