Qd(p)-FREE RANK TWO FINITE GROUPS ACT FREELY ON A HOMOTOPY PRODUCT OF TWO SPHERES

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Abstract. A classic result of Swan states that a finite group G acts freely on a finite homotopy sphere if and only if every abelian subgroup of G is cyclic. Following this result, Benson and Carlson conjectured that a finite group G acts freely on a finite complex with the homotopy type of n spheres if the rank of G is less than or equal to n. Recently, Adem and Smith have shown that every rank two finite p-group acts freely on a finite complex with the homotopy type of two spheres. In this paper we will make further progress, showing that rank two groups which are Qd(p)-free act freely on a finite homotopy product of two spheres.

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1. Introduction

Recall that the p-rank of a finite group G, $\mathrm{rk}_p(G)$, is the largest rank of an elementary abelian p-subgroup of G and that the rank of a finite group G, $\mathrm{rk}(G)$, is the maximum of $\mathrm{rk}_p(G)$ taken over all primes p. In addition we can define the homotopy rank of a finite group G, $\mathrm{h}(G)$, to be the minimal integer k such that G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2} \times \cdots \times \mathbb{S}^{n_k}$. Benson and Carlson [7] state a conjecture that for any finite group G, $\mathrm{rk}(G) = \mathrm{h}(G)$. The case of this conjecture when G is a rank one group is a direct result of Swan's theorem [28]. Benson and Carlson's conjecture has also been verified by Adem and Smith [1, 2] in the case when G is a rank two p-group. In addition, recall that Heller [19] has shown that for a finite group G, if $\mathrm{h}(G) = 2$, then $\mathrm{rk}(G) = 2$. In this paper we will verify this same conjecture for all rank two groups that do not contain a particular type of subquotient.

Before we state our main theorem, we make a few definitions.

Definition 1. A *subquotient* of a group G is a factor group H/K where $H, K \subseteq G$ with $K \lhd H$. A subgroup L is said to be involved in G if L is isomorphic to a subquotient of G. In particular, for a prime p, we say that L is p'-involved in G if L is isomorphic to a subquotient H/K of G where K has order relatively prime to p.

Definition 2. Let p be a prime and E be a rank two elementary abelian p-group. E is then also a two-dimensional vector space over $\mathbb{F}_p$. We define the quadratic group Qd(p) to be the semidirect product of E by the special linear group $SL_2(\mathbb{F}_p)$.

A classic result of Glauberman shows that if for some odd prime p $\mathrm{Qd}(p)$ is not involved in a finite group G then the p-fusion is controlled by the normalizer of a characteristic p-subgroup [13, 14]. Although we do not directly use Glauberman's result there appears to be a connection as p-fusion plays a integral role in our investigation. We can now state our main theorem.

Theorem 3. *Let G be a finite group such that $\mathrm{rk}(G) = 2$. G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$ unless for some odd prime p, $Qd(p)$ is p'-involved in G.*

The following corollary, which can also be shown using more direct methods, follows immediately from Theorem 3:

Corollary 4. *If G is a finite group of odd order with $\mathrm{rk}(G) = 2$, then G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{n_1} \times \mathbb{S}^{n_2}$.*

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2. Preliminaries

Recall that Adem and Smith have shown that if G is a rank two p-group then $h(G) = 2$ [1, 2]. In their proof they use the following theorem:

Theorem 5 (Adem and Smith [2])**.** *Let G be a finite group and let X be a finitely dominated, simply connected G-CW complex such that every isotropy subgroup has rank one. Then for some integer $N > 0$ there exists a finite CW-complex $Y \simeq \mathbb{S}^N \times X$ and a free action of G on Y such that the projection $Y \to X$ is G-equivariant.*

In the present work, we apply Theorem 5 for a rank two group by finding an action of the group on a CW-complex $X \simeq S^n$ such that the isotropy subgroups are of rank one. Adem and Smith in [2] also developed a sufficient condition for this type of action to exist.

To explore this sufficient condition, we start by letting G be a finite group. Given a G-CW complex $X \simeq \mathbb{S}^{N-1}$, there is a fibration $X \to X \times_G EG \to BG$. This fibration gives an Euler class β, which is a cohomology class in $H^N(G)$. This cohomology class is the transgression in the Serre spectral sequence of the fundamental class of the fiber. Regarding this class Adem and Smith made the following definition:

Definition 6 (Adem and Smith [2])**.** Let G be a finite group. The Euler class β of a G-CW complex $X \simeq \mathbb{S}^{N-1}$ is called effective if the Krull dimension of $H^*(X \times_G EG, \mathbb{F}_p)$ is less than $\mathrm{rk}(G)$ for all $p||G|$.

Adem and Smith have proven the following lemma characterizing effective Euler classes:

Lemma 7 (Adem and Smith [2])**.** *Let $\beta \in H^N(G)$ be the Euler class of an action on a finite dimensional $X \simeq \mathbb{S}^{N-1}$. β is effective if and only if every maximal rank elementary abelian subgroup of G acts without stationary points.*

Remark 8. By applying Lemma 7 and Theorem 5, we see that any finite group of rank two having an effective Euler class of an action on a finite dimensional $X \simeq \mathbb{S}^{N-1}$ acts freely on a finite CW-complex $Y \simeq \mathbb{S}^m \times \mathbb{S}^{N-1}$.

We now make a topological definition which will allow us to classify certain actions on homotopy spheres.

Definition 9. Consider the set of maps from a space X to a space Y. Two such maps f and g are called homotopic if there exists a continuous function $H : X \times [0,1] \to Y$ such that for all $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. This induces an equivalence relation on the set of maps from X to Y given by two maps are equivalent if they are homotopic. The resulting equivalence classes are called homotopy classes of maps from X to Y and the set of such classes is denoted $[X,Y]$.

Recall that for a group G, BG is the classifying space of G. Also $[BG, BU(n)]$ represents the homotopy classes of maps from BG to $BU(n)$, which may also be written as $\pi_0\mathrm{Hom}(BG, BU(n))$ (see [12]). In addition, recall that $[BG, BU(n)]$ is isomorphic to the set of isomorphism classes of rank n complex vector bundles over BG. In this discussion, complex vector bundles over BG will be thought of as elements of $[BG, BU(n)]$.

Recall that the Euler class of the universal bundle over $BU(n)$ is the top Chern class, which lies in $H^{2n}(BU(n))$. We will call this class ξ. A map $\varphi : BG \to BU(n)$ induces a map $\varphi^* : H^{2n}(BU(n)) \to H^{2n}(BG) = H^{2n}(G)$. Thus the image of ξ under φ^* is the Euler class of the vector bundle over BG that corresponds to φ. Let $\beta_\varphi = \varphi^*(\xi)$ be this class, which is the Euler class of a G-CW complex that is the unit sphere bundle of the vector bundle. This G-CW complex is the fiber as discussed at the outset. Recall that if φ and μ are homotopic maps from BG to $BU(n)$, then $\beta_\varphi = \beta_\mu$; therefore, the Euler class of a homotopy class of maps is well defined.

Recall that G is a finite group, p is a prime dividing the order of G, and G_p is a Sylow p-subgroup of G. In addition we will use $\mathrm{Char}_n(G_p)$ to represent the set of degree n complex characters of G_p and will let $\mathrm{Char}_n^G(G_p)$ represent the subset consisting of those degree n complex characters of G_p that are the restrictions of class functions of G. Also we use the notation $\mathrm{Rep}(G, U(n))$ for the degree n unitary representations of the group G. This can be viewd as the following equality $\mathrm{Rep}(G, U(n)) = \mathrm{Hom}(G, U(n))/\mathrm{Inn}(U(n))$ where we mod out by the inner automorphisms of $U(n)$. Recall that Dwyer and Zabrodsky [12] have shown that for any p-subgroup P,

$$\mathrm{Char}_n(P) \cong \mathrm{Rep}(P, U(n)) \xrightarrow{\cong} [BP, BU(n)]$$

where the map is $\rho \mapsto B\rho$. So if φ is a map $BG \to BU(n)$ and if P is a p-subgroup of G for some prime p, $\varphi|_{BP}$ is induced by a unitary representation of P. The following proposition gives the characterization of effectiveness for the Euler class β_φ.

Proposition 10 ([1, section 4])**.** *An Euler class β_φ is effective if and only if for each elementary abelian subgroup E of G with $\mathrm{rk}(E) = \mathrm{rk}(G)$, there is a unitary representation $\lambda : E \to U(n)$ such that $\varphi_E = B\lambda$ and that λ does not have the trivial representation as one of its irreducible constituents.*

As we proceed we will want to work separately with each prime. Definition 11 will be useful in this endeavor.

Definition 11. Let G be a finite group. The Euler class β_φ is called p-effective if for each elementary abelian p-subgroup E of G with $\mathrm{rk}(E) = \mathrm{rk}(G)$, there is a unitary representation $\lambda : E \to U(n)$ such that $\varphi_E = B\lambda$ and that λ does not have the trivial representation as one of its irreducible constituents.

Remark 12. It is clear then that β_φ is effective if and only if it is p-effective for each $p||G|$. Also if $\mathrm{rk}_p(G) < \mathrm{rk}(G)$, then any β_φ is p-effective since it trivially satisfies Definition 11; therefore, we can restate Proposition 10: β_φ is effective if and only if it is p-effective for each prime p such that $\mathrm{rk}_p(G) = \mathrm{rk}(G)$.

Definition 13 concerns the characters of subgroups of G.

Definition 13. Let $H \subseteq G$ be a subgroup. Let χ be a character of H. We say that χ *respects fusion in* G if for each pair of elements h, $k \in H$ such that $\exists\, g \in G$ with $h = gkg^{-1}$, $\chi(h) = \chi(k)$.

Remark 14. Let $H \subseteq G$. A sufficient condition for a character of H to respect fusion in G is for the character to be constant on elements of the same order.

Notice that for $\chi \in \mathrm{Char}_n(G_p)$, χ respects fusion in G if and only if $\chi \in \mathrm{Char}_n^G(G_p)$. In a previous paper [23], the author has shown the following theorem.

Theorem 15 (Jackson [23, Theorem 1.3])**.** *If G is a finite group that does not contain a rank three elementary abelian subgroup, then the natural mapping*

$$\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p)$$

is a surjection.

Combining Theorem 15 and Definition 11, we get the following theorem.

Theorem 16. *Let G be a finite group with $\mathrm{rk}(G) = 2$, p a prime dividing $|G|$, and G_p a Sylow p-subgroup of G. If there is a character χ of G_p that respects fusion in G and has the property that $[\chi|_E, 1_E] = 0$ for each $E \subseteq G_p$ elementary abelian with $\mathrm{rk}(E) = 2$, then there is a complex vector bundle over BG whose Euler class is p-effective.*

Remark 17. Let p be a prime, G_p a p-group of rank n, and χ a character of G_p. A sufficient condition to guarantee that $[\chi|_E, 1_E] = 0$ for each $E \subseteq G_p$ elementary abelian with $\mathrm{rk}(E) =$

$\mathrm{rk}(G_p)$, is that $\chi(1) = c(p^n - 1)$ and $\chi(g) = -c$ for each element of G_p of order p. In this case, $\chi|_E = c(\rho - 1)$ where ρ is the regular character of the group E.

Theorem 16 leads to definition 18:

Definition 18. Let G be a finite group, p a prime dividing $|G|$, and G_p a Sylow p-subgroup of G. A character χ of G_p is called a *p-effective character of G* under two conditions: χ respects fusion in G, and for each $E \subseteq G_p$ elementary abelian with $\mathrm{rk}(E) = \mathrm{rk}(G)$, $[\chi|_E, 1_E] = 0$.

Remark 19. Let G be a finite group, p a prime dividing $|G|$, and G_p a Sylow p-subgroup of G. A character χ of G_p is p-effective if χ is both constant on elements of the same order and $[\chi|_E, 1_E] = 0$ for each $E \subseteq G_p$ elementary abelian with $\mathrm{rk}(E) = \mathrm{rk}(G)$. (This follows from Remark 14.)

Applying Definition 18, Theorem 16 can be restated as follows: If G has a p-effective character, then there is a complex vector bundle over BG whose Euler class is p-effective.

We have now essentially proved Theorem 20, which is a modified form of a theorem by Adem and Smith.

Theorem 20 (See [2, Theorem 7.2])**.** *Let G be a finite group with $\mathrm{rk}(G) = 2$. If for each prime p dividing the order of G there exists a p-effective character of G, then G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{N_1} \times \mathbb{S}^{N_2}$.*

Proof. For each prime p dividing the order of G, let χ_p be a p-effective character of G. There exists an integer n such that $\chi_p(1)$ divides n for each p dividing the order of G.

$$(\frac{n}{\chi_{p_1}(1)} \cdot \chi_{p_1}, \frac{n}{\chi_{p_2}(1)} \cdot \chi_{p_2}, \ldots, \frac{n}{\chi_{p_k}(1)} \cdot \chi_{p_k}) \in \prod_{p||G|} \mathrm{Char}_n^G(G_p).$$

By Theorem 15, there exists an element $\varphi \in [BG, BU(n)]$ such that

$$\psi_G(\varphi) = (\frac{n}{\chi_{p_1}(1)} \cdot \chi_{p_1}, \frac{n}{\chi_{p_2}(1)} \cdot \chi_{p_2}, \ldots, \frac{n}{\chi_{p_k}(1)} \cdot \chi_{p_k}).$$

The Euler class of the homotopy class φ is then an effective Euler class of G. Theorem 20 then follows from Remark 8. □

Theorem 20 reduces the problem of showing that a rank two finite group G acts freely on a finite CW-complex $Y \simeq \mathbb{S}^{N_1} \times \mathbb{S}^{N_2}$ to the problem of demonstrating that for each prime p there is a p-effective character of G. It is important to state that not all rank two finite groups have a p-effective character for each prime p. An example of a finite group that does not contain a p-effective character for a particular prime is given in the following remark, which will be proven in Section 6.

Remark 21. Let p be an odd prime. The group $PSL_3(\mathbb{F}_p)$ does not have a p-effective character.

We mention that the groups $PSL_3(\mathbb{F}_p)$, for odd primes p, are the only finite simple groups of rank 2 that were not shown by Adem and Smith to act freely on a finite CW-complex $Y \simeq \mathbb{S}^{N_1} \times \mathbb{S}^{N_2}$ [1, 2].

3. A sufficient condition for p-effective characters

In Section 3, we establish a sufficient condition for the existence of a p-effective character. We start by recalling the following definitions.

Definition 22. Let G be a finite group and H and K subgroups such that $H \subset K$. We say that H is strongly closed in K with respect to G if for each $g \in G$, $gHg^{-1} \cap K \subseteq H$.

Remark 23. Let G be a finite group and H and K subgroups such that $H \subset K$. We say that H is weakly closed in K with respect to G if for each $g \in G$ with $gHg^{-1} \subseteq K$, $gHg^{-1} = H$. For a subgroup H of prime order, H is strongly closed in K with respect to G if and only H weakly closed in K with respect to G.

Definition 24. Let P be a p-group. We us $\Omega_1(P)$ to denote p-subgroup of P generated by all of the order p-elements of P. In particular, if P is abelian, $\Omega_1(P)$ is elementary abelian, and if P is cyclic, then $\Omega_1(P)$ has order p.

Lemma 25. *If E is an elementary abelian subgroup of a p-group P, then $\langle E, \Omega_1(Z(P))\rangle$ is an elementary abelian subgroup of P. In particular, if E is a maximal elementary abelian subgroup of P under inclusion, then $\Omega_1(Z(P)) \subseteq E$.*

Proof. Let $Z = \Omega_1(Z(P))$. Since Z is central, the subgroup $\langle E, \Omega_1(Z(P))\rangle$ is abelian. The product is generated by elements of order p and so must be elementary abelian. □

Corollary 26. *Let P be a p-group with $\mathrm{rk}(P) = \mathrm{rk}(Z(P))$. $\Omega_1(Z(P))$ is the unique elementary abelian subgroup of P that is maximal under inclusion. In addition, $\Omega_1(P) = \Omega_1(Z(P))$, and $\Omega_1(Z(P))$ is strongly closed in P with respect to G.*

Proof. If E is a maximal elementary abelian subgroup of P under inclusion, then $\Omega_1(Z(P)) \subseteq E$ by Lemma 25. Equality holds since $\mathrm{rk}(E) \leq \mathrm{rk}(P) = \mathrm{rk}(\Omega_1(Z(P)))$. □

Again we let p be a prime dividing the order of a finite group G and let $G_p \in \mathrm{Syl}_p(G)$. We show that if there is a subgroup of $Z(G_p)$ that is strongly closed in G_p with respect to G, then G has a p-effective character in the following proposition:

Proposition 27. *Let G be a finite group, $n = \mathrm{rk}(G)$, p a prime divisor of $|G|$ with $\mathrm{rk}_p(G) = n$, and G_p a Sylow p-subgroup of G. If there exists $H \subseteq Z(G_p)$ such that H is non-trivial and strongly closed in G_p with respect to G, then G has a p-effective character.*

Before we prove Proposition 27, we state the following standard lemma, which we will need in the proof of Proposition 27.

Proof of Proposition 27. Let $H \subseteq Z(G_p)$ such that H is non-empty and strongly closed in G_p with respect to G. Let $K = \Omega_1(H)$. So there is an $i \leq n$ with $K \cong (\mathbb{Z}/p\mathbb{Z})^i$. Let $\chi = \sum_{\psi \in Irr(K)\setminus\{1_K\}} \psi$; therefore, $[\chi, 1_K] = 0$. Let $\varphi = \mathrm{Ind}_K^{G_p}\chi$.
Notice that for each $y \in K \setminus \{e\}$, $\chi(y) = -1$ [16, 4.2.7.ii]. In addition since $K \subseteq Z(G_p)$, for each $y \in K \setminus \{e\}$, $\varphi(y) = -[G_p : K]$ [16, 4.4.3.i]. Since H is strongly closed in G_p with respect to G, K is strongly closed in G_p with respect to G. φ respects fusion in G; therefore, for each $z \in G_p \setminus K$, $\varphi(z) = 0$.

Let E be a subgroup of G_p such that $E \cong (\mathbb{Z}/p\mathbb{Z})^n$. By Lemma 25, $K \subseteq E$ since $K \subseteq \Omega_1(Z(G_p))$. Now suppose $\exists\ E \subseteq G_p$ such that $E \cong (\mathbb{Z}/p\mathbb{Z})^n$ and $[\varphi|_E, 1_E] > 0$. $K \subseteq \Omega_1(Z(G_p))$ implies that $K \subseteq E$; therefore, clearly, $[(\varphi|_E)|_K, 1_K] > 0$. Note that $(\varphi|_E)|_K = \varphi|_K$ and that $\varphi|_K = [G_p : K]\chi$; thus $[\varphi|_K, 1_K] = [G_p : K][\chi, 1_K] = 0$. This is a contradiction showing that if E is an elementary abelian p-subgroup of G_p of rank n, $[\varphi|_E, 1_E] = 0$. φ is therefore a p-effective character of G. □

We also prove a Corollary to Proposition 27.

Corollary 28. *Let G be a finite group, p a prime divisor of $|G|$ with $\mathrm{rk}(G) = \mathrm{rk}_p(G) = n$, and G_p a Sylow p-subgroup of G. If $\mathrm{rk}(Z(G_p)) = n$, then G has a p-effective character.*

Proof. By Corollary 26, $\Omega_1(Z(G_p))$ is strongly closed in G_p with respect to G. Corollary 28 then follows from Proposition 27. □

In Section 4 and 5, we will look at the possible types of rank two groups where this sufficient condition holds. In Section 4 we will look at the case where p is an odd prime, while in Section 5 we will examine the case where $p = 2$.

4. Odd primes

When p is an odd prime and $\mathrm{rk}_p(G) = 2$, we show that if the sufficiency condition of Proposition 27 does not hold, then Qd(p) must be p'-involved in G.

Before we list the next theorem, we need two definitions and a lemma.

Definition 29. Let $P \subseteq G$ be a p-subgroup. P is said to be *p-centric* if $Z(P)$ is a Sylow p-subgroup of $C_G(P)$. Equivalently P is p-centric if and only if $C_G(P) = Z(P) \times C'_G(P)$ where $C'_G(P)$ is a p'-subgroup.

Definition 30 (see [17])**.** Let $P \subseteq G$ be a p-centric subgroup. P is said to be *principal p-radical* if $O_p(N_G(P)/PC_G(P)) = \{1\}$.

Lemma 31 (Diaz, Ruiz, Viruel [10, Section 3])**.** *Let G be a finite group and $p > 2$ a prime with $\mathrm{rk}_p(G) = 2$. Let $G_p \in \mathrm{Syl}_p(G)$ and P be a principal p-radical subgroup of G. If $P \neq G_p$ and P is metacyclic, then $P \cong \mathbb{Z}/_{p^n} \times \mathbb{Z}/_{p^n}$ with either $p = 3$ or $n = 1$.*

The lemma above follows from Corollaries 3.7 and 3.8 as well as Proposition 3.12 of [10] . The argument in [10] is written in the language of fusion systems, but can be applied to finite groups giving Lemma 31. We will also make use of the following well known lemma, which appears in [10].

Lemma 32 ([10, Lemma 3.11])**.** *Let G be a subgroup of $GL_2(p)$ for an odd prime p. If $O_p(G) = \{1\}$ and p divides the order of G, then $SL_2(p) \subseteq G$.*

Lemma 33. *Let H be a finite group and p an odd prime such that $O_{p'}(H) = 1$, $O_p(H)$ is homocyclic abelian of rank two and also not a Sylow p-subgroup of H. If $C_H(O_p(H)) \subseteq O_p(H)$, then H has a subgroup isomorphic to* $\mathrm{Qd}(p)$.

Proof. Let $P = O_p(H) = \mathbb{Z}/_{p^n} \times \mathbb{Z}/_{p^n}$. We may consider the group of automorphisms $\mathrm{Aut}(\mathbb{Z}/_{p^n} \times \mathbb{Z}/_{p^n})$ as two by two matrices with coefficients in $\mathbb{Z}/_{p^n}$ and with p not dividing the determinant. Reduction modulo p gives the following short exact sequence of groups:

$$1 \to Q \to \mathrm{Aut}(\mathbb{Z}/_{p^n} \times \mathbb{Z}/_{p^n}) \xrightarrow{\rho} \mathrm{GL}_2(\mathbb{F}_p) \to 1$$

with Q a p-group. Notice that $H/P \subseteq \mathrm{Aut}(\mathbb{Z}/_{p^n} \times \mathbb{Z}/_{p^n})$. Since $O_p(H) = P$, $Q \cap H/P = \{1\}$. And so ρ is an injection when restricted to H/P. Notice that $\rho(H/P)$ has order divisible by p, since P is not a Sylow p-subgroup of H; therefore, by Lemma 32, $\mathrm{SL}_2(p) \subseteq \rho(H/P)$. So a restriction of ρ is now an isomorphism between a subgroup of K of H/P and $SL_2(\mathbb{F}_p)$. Let $\alpha \in K$ be preimage under ρ of the scalar matrix with -1 diagonal entries in $SL_2(\mathbb{F}_p)$. Let π be the quotient map $H \to H/P$. We see that α has a preimage under π which has order 2. Call this preimage β. P can be viewed as two-dimensional vector space over $\mathbb{F}_{p^n}$ where p^n is the exponent of P. Notice that β acts on elements of P by scalar multiplication by -1 and so has no nontrivial fixed points in P. Let C be the centralizer of β in H. Since $C \cap P = \{1\}$, $\pi|_C$ is an isomorphism. Also since K is contained in the centralizer of $\alpha = \pi(\beta)$ in H/P, $K \subseteq \pi(C)$. It follows that C has a subgroup L isomorphic to K and hence isomorphic to $SL_2(\mathbb{F}_p)$. So $\langle L, \Omega_1(\bar{P})\rangle$ is a subgroup of H which is isomorphic to $\mathrm{Qd}(p)$. □

Theorem 34. *Let G be a finite group and $p > 2$ a prime with $\mathrm{rk}_p(G) = 2$. Let $G_p \in \mathrm{Syl}_p(G)$. If $\Omega_1(Z(G_p))$ is not strongly closed in G_p with respect to G, then $\mathrm{Qd}(p)$ is p'-involved in G.*

Proof. Notice that $\mathrm{rk}(Z(G_p)) = 1$ by Corollary 26. Let $Z = \Omega_1(Z(G_p))$. By hypothesis, Z is not strongly closed in G_p with respect to G. Recall that Alperin's Fusion Theory [3] states that given a weak conjugation family $\mathcal{F}$, there exists an $(S, T) \in \mathcal{F}$ with $Z \subseteq S \subseteq G_p$ and with Z not strongly closed in S with respect to T. The collection of all pairs $(P, N_G(P))$, where $P \subseteq G_p$ is a principal p-radical subgroup of G, is the weak conjugation family of Goldschmidt [15, Theorem 3.4] (see also [23]). So there exists $P \subset G_p$ that is a principal p-radical subgroup of G with $Z \subseteq P$ and with Z not strongly closed in P with respect to $N_G(P)$. Notice that $Z \neq P \neq G_p$ and that $\mathrm{rk}_p(Z(P)) = \mathrm{rk}_p(P) = 2$ (in particular $p|[N_G(P) : P]$). Using the classification of rank two p-groups for odd primes p by Blackburn [8] (see also [10, 11]), we notice that P must be metacyclic (since $\mathrm{rk}_p(Z(P)) = 2$). By Lemma 31, $P \cong \mathbb{Z}/_{p^n} \times \mathbb{Z}/_{p^n}$

with either $p = 3$ or $n = 1$. Now consider the group $H = N_G(P)/C'_G(P)$. We want to see that H satisfies the conditions of Lemma 33. Let $\bar{P} = (PC'_G(P))/C'_G(P)$. First $O_p(H) = \bar{P}$ since P is principal p-radical. Next let $K = O_{p'}(H)$. Since K and P are both normal in H with trivial intersection, $K \subseteq C_H(P)$; therefore $K = \{1\}$. Since $P \neq G_p$, $\bar{P}$ is not a Sylow p-subgroup of H. Lastly since P is p-centric and no p' elements of H commute with P, $C_H(P) \subseteq P$. Now applying Lemma 33, we obtain that H has a subgroup Q isomorphic to Qd(p). Thus Q is a subquotient of G obtained by taking a p' quotient and so Qd(p) is p'-involved in G. □

We point out here some connections between the results of Theorem 34 and previous work by Adem and Smith. Notice that Theorem 34 together with Proposition 27 implies that any rank 2 group G that does not have Qd(p) p'-involved in G has a p-effective character. A p-group P is called a Swan group if for any group G containing P as a Sylow p-subgroup, the mod-p cohomology ring $H^*(G)$ is equal to $H^*(N_G(P)) \cong H^*(P)^{N_G(P)}$. A group with a Sylow p-subgroup that is a Swan group has been shown to have a p-effective character by Adem and Smith [2, Section 6]. Dietz and Glauberman (see [25]) have shown that any odd order metacyclic group is a swan group. In addition, Diaz, Ruiz and Viruel [10] have shown that if P is a rank 2 p-group for $p > 3$, P is Swan unless it is isomorphic to a Sylow p-subgroup of Qd(p). They also demonstrate that some rank two 3-groups of maximal class are Swan.

5. Prime two

The argument in this section will be done in two parts. First we will show that for the prime two, if the sufficiency condition of Proposition 27 does not hold for G and for prime 2, then $G_2 \in \mathrm{Syl}_2(G)$ is either dihedral, semi-dihedral, or wreathed. In the second part we will deal with these exeptional cases. We start with two definitions:

Definition 35 ([16, p. 191])**.** Recall that a 2-group is semi-dihedral (sometimes called quasi-dihedral) if it is generated by two elements x and y subject to the relations that $y^2 = x^{2^n} = 1$ and $yxy^{-1} = x^{-1+2^{n-1}}$ for some $n \geq 3$.

Definition 36 ([16, p. 486])**.** A 2-group is called wreathed if it is generated by three elements x, y, and z subject to the relations that $x^{2^n} = y^{2^n} = z^2 = 1$, $xy = yx$, and $zxz^{-1} = y$ with $n \geq 2$.

Notice that the dihedral group of order 8 is wreathed. Now that we have these definitions we will state Proposition 37, which is crucial in our discussion of rank two 2-subgroups. The proof of Proposition 37 follows the proof of a theorem by Alperin, Brauer, and Gorenstein [5, Proposition 7.1].

Proposition 37 (See Proposition 7.1 of [5] and its proof.)**.** *Let G be a finite group and $G_2 \in \mathrm{Syl}_2(G)$. Suppose that* $\mathrm{rk}(G_2) = 2$. *If $\Omega_1(Z(G_2))$ is not strongly closed in G_2 with respect to G, then G_2 is either dihedral, semi-dihedral, or wreathed.*

Since this proof follows the proof of Proposition 7.1 of [5], we will only state a brief outline, including the changes needed to prove our Proposition 37.
Again notice that $\mathrm{rk}(Z(G_2)) = 1$ by Corollary 26. If G_2 does not contain a normal rank 2 elementary abelian subgroup, then G_2 must be either dihedral or semi-dihedral (see [16, Theorem 5.4.10]). We may assume that G_2 has a normal rank 2 elementary abelian subgroup, which we will call V. Letting $T = C_{G_2}(V)$, we notice that $[G_2 : T] \leq 2$. Also $G_2 \neq T$ since $\mathrm{rk}(Z(G_2)) = 1$; therefore, $[G_2 : T] = 2$.

Letting $A_G(V) = N_G(V)/C_G(V)$, we notice that $A_G(V)$ is isomorphic to a subgroup of $\mathrm{Aut}(V) \cong \Sigma_3$ and that $2 \mid |A_G(V)|$. Thus either $A_G(V) \cong \Sigma_3$ or $2 = |A_G(V)|$.

From the argument in [5] we get the following results: if $A_G(V) \cong \Sigma_n$, then G_2 is wreathed (including the case of D_8); and if $|A_G(V)| = 2$, then $G_2 \cong D_8$. This finishes the discussion of Proposition 37.

We have seen that if G is a rank two finite group, it has a 2-effective character unless a Sylow 2-subgroup is dihedral, semi-dihedral, or wreathed. In the rest of this section we will show that in each of these three cases G also has a 2-effective character, which will complete the proof that a rank two finite group has a 2-effective character. The author proved the following three lemmas in his doctoral thesis [24]; the proofs are included here for completeness. We will start with a lemma that will be used in the proof of Proposition 41.

Lemma 38. *Let G be a finite group, let $n = \mathrm{rk}(G)$, let p a prime divisor of $|G|$ with $\mathrm{rk}_p(G) = n$, and let $G_p \in \mathrm{Syl}_p(G)$. If G_p is abelian, $G_p \lhd G$, or G is p-nilpotent, then G has a p-effective character.*

Proof. In each case we will show that G has a p-effective character by showing that $Z(G_p)$ is strongly closed in G_p with respect to G using Proposition 27. If G_p is abelian, $Z(G_p) = G_p$ is obviously strongly closed in G_p with respect to G (see Corollary 26). In the case where $G_p \lhd G$, $Z(G_p) \lhd G$ because $Z(G_p)$ is a characteristic subgroup of G_p; therefore, $Z(G_p)$ is strongly closed in G_p with respect to G. If G is p-nilpotent then two elements of G_p are conjugate in G if and only if they are conjugate in G_p. Thus, in this case also, $Z(G_p)$ is strongly closed in G_p with respect to G. □

Lemma 39. *If P is a dihedral or semi-dihedral 2-group such that $|P| = 2^n$ with $n \geq 3$, then there is a character χ of P such that*

$$\chi(g) = \begin{cases} 3 * 2^{n-3} & \text{if } g = 1 \\ -2^{n-3} & \text{if } g \text{ is an involution} \\ 2^{n-3} & \text{otherwise.} \end{cases}$$

Proof. Let N be the commutator subgroup $[P, P]$ and notice that $P/N \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. There is a non-trivial irreducible character λ of P/N with $\lambda(gN) = 1$ for $g \in Z(P)$ (recall $Z(P) \cong \mathbb{Z}/2$)

and such that If $g \in P \setminus Z(P)$ is an involution, then $\lambda(gN) = -1$. Let $\varphi(g) = \lambda(gN)$, which is a character of P.

We will define another character of P by first inductively defining a character of each D_{2^m}. We start by letting ψ_3 be the character of D_8 such that

$$\psi_3(g) = \begin{cases} 2 & \text{if } g = 1 \\ -2 & \text{if } g \in Z(D_8) \setminus \{1\} \\ 0 & \text{otherwise.} \end{cases}$$

For the induction, let $\psi_k = \mathrm{Ind}_{D_{2^{k-1}}}^{D_{2^k}} \psi_{k-1}$. ψ_k is a character of D_{2^k} for each $k \geq 3$. We see that if P is dihedral, then $\chi = 2^{n-3}\varphi + \psi_n$, and Lemma 39 holds for dihedral 2-groups. On the other hand, if P is semidihedral, $\chi = 2^{n-3}\varphi + \mathrm{Ind}_{D_{2^{n-1}}}^{P} \psi_{n-1}$. □

Lemma 40. *Let G be a finite group with $S \in \mathrm{Syl}_2(G)$ a wreathed 2-group. If $\mathrm{rk}(G) = 2$ and G has no normal subgroup of index 2, then G has a 2-effective character.*

Proof. Let S be a Sylow 2-subgroup of G. We will say that S is generated by x, y, and z as in Definition 36. Let α be any primitive $(2^n)^{th}$ root of unity. We will define a degree 3 complex representation of S, as follows. $\kappa : S \to GL_3(\mathbb{C})$ is given by:

$$\kappa(x) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha^{-2} \end{pmatrix}$$

$$\kappa(y) = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \alpha^{-2} & 0 \\ 0 & 0 & \alpha \end{pmatrix}$$

$$\kappa(z) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Let ν be the complex character of S associated to κ (i.e. for each $g \in S$, $\nu(g) = \mathrm{tr}(\kappa(g))$. We will now show that ν is a 2-effective character for G.

First notice the character values on involutions. The elements of S which are involutions are $x^{2^{n-1}}$, $y^{2^{n-1}}$, $(xy)^{2^{n-1}}$, and those conjugate in S to z. Clearly ν takes the value -1 for each of these elements. Therefore if H is a rank 2 elementary abelian subgroup of S, then $[\nu|_H, 1_H] = 0$.

Let S_0 be the subgroup of S generated by x and y. S_0 is an abelian normal subgroup of S of index 2. Before we go on we point out that z is conjugate to an element of S_0, in particular to $x^{2^{n-1}}$, and $[N_G(S_0) : C_G(S_0)] = 6$ (see [9, Proposition 2G*]). Now we look at fusion of elements of S_0. Any two elements of S_0 that are conjugate in G, are conjugate in

$N_G(S_0)$ (see [9, Proposition 2D]). $N_G(S_0)$ is generated by $C_G(S_0)$, the elements z, and an element $\eta \in G$ such that $\eta x \eta^{-1} = y$, $\eta y \eta^{-1} = x^{-1}y^{-1}$, and $\eta(x^{-1}y^{-1})\eta^{-1} = x$ [9, Proposition $2\mathrm{B_I}$].

To show that ν respects fusion of elements in S_0, it is enough to show that if $g \in S_0$, then $\nu(\mu) = \nu(zgz^{-1}) = \nu(\eta g \eta^{-1})$ (assuming that such an η exists). Notice that for any $g \in S_0$, there are integers k and l with $g = x^k y^l$. We see then that $zgz^{-1} = x^l y^k$ and $\nu(x^k y^l) = \alpha^{k+l} + \alpha^{k-2l} + \alpha^{-2k+l} = \nu_i(x^l y^k)$. Also $\eta g \eta^{-1} = x^{-k}y^{l-k}$ and $\nu(x^k y^l) = \alpha^{k+l} + \alpha^{k-2l} + \alpha^{-2k+l} = \alpha^{l-k-k} + \alpha^{-k-2(l-k)} + \alpha^{2k+l-k} = \nu(x^{-k}y^{l-k})$.

Next we will check fusion in elements of order at most 2^n. If $g \in S \setminus S_0$ is of order at most 2^n, then there is an integer h with g conjugate to $x^{-h}y^{-h}z$ in S (see [9, Lemma 4]). In particular $\nu(g) = \nu(x^{-h}y^{-h}z)$. Now $x^{-h}y^{-h}z$ is conjugate to $x^{2^{n-1}-h}y^{-h}$ in G (see [9, Proposition $2\mathrm{F_a}$]). We see that $\nu(x^{-h}y^{-h}z) = -\alpha^{-2h} = \alpha^{2^{n-1}}\alpha^{-2h} = \nu(x^{2^{n-1}-h}y^{-h})$.

At this point we have finished the fusion of elements of order at most 2^m. Notice that all elements of order larger than 2^m must have order 2^{m+1} and must be in $S \setminus S_0$. Suppose that $g \in S$ is of order 2^{m+1} and is conjugate to another element g' of S. It immediately follows that $g, g' \in S \setminus S_0$ and that g and g' are conjugate in S (see [9, Proposition 2E]). So any character of S, in particular ν, must agree on g and g'. This finishes the proof of the Lemma 40. □

Proposition 41 (Jackson [24])**.** *If G is a finite group with a dihedral, semi-dihedral, or wreathed Sylow 2-subgroup such that* $\mathrm{rk}(G) = 2$*, then G has a 2-effective character.*

Proof. Let $G_2 \in \mathrm{Syl}_2(G)$ and let n be the integer with $|G_2| = 2^n$. If G_2 is either dihedral or semi-dihedral, then there is a character of G_2 given in Lemma 39 such that

$$\chi(g) = \begin{cases} 3 * 2^{n-3} & \text{if } g = 1 \\ -2^{n-3} & \text{if } g \text{ is an involution} \\ 2^{n-3} & \text{otherwise .} \end{cases}$$

Since χ is constant on elements of the same order, it must respect fusion in G as in Remark 19. It is also clear that if $E \subseteq G_2$ is a rank 2 elementary abelian subgroup, $[\chi|_E, 1_E] = 0$ (see Remark 17). So χ is a 2-effective character of G.

We may now assume that G_2 is wreathed and is generated by x, y, and z as in Definition 36. In addition, let S_0 be the abelian subgroup of G_2 generated by x and y. Since G_2 is wreathed, one of the following holds (see [4, Proposition 2, p. 11]):

(1) G has no normal subgroups of index 2,
(2) $Z(G_2)$ is strongly closed in G_2 with respect to G,
(3) G has a normal subgroup K with Sylow 2-subgroup S_0 generated by x and y, or

(4) G is 2-nilpotent.

Case 1 was treated in Lemma 40. Case 2 has been treated in Proposition 27.

We now treat case 3. Suppose G has a normal subgroup K with Sylow 2-subgroup S_0 generated by x and y. Notice that $G_2 \cap K = S_0$. We see that S_0 is weakly closed in G_2 with respect to G. By a standard result of Burnside (see [6, 37.6]), $N_G(S_0)$ controls fusion in $C_{G_2}(S_0) = S_0$. Also notice that any element of $G_2 \setminus S_0$ is conjugate in G to only those elements to which it is conjugate in G_2. Using the Frattini factor subgroup of S_0, we notice that $N_G(S_0)/C_G(S_0)$ is isomorphic to a subgroup of Σ_3 (see [9, Page 264] or [4, Page 12]). We may assume that $Z(G_2)$ is not strongly closed in G_2 with respect to G, otherwise the group has been treated in case 2. In particular, this implies that $\Omega_1(Z(G_2)) = \langle (xy)^{2^{n-1}} \rangle$ is not strong closed in G_2 with respect to G. The involution $(xy)^{2^{n-1}}$ must then be conjugate in $N_G(S_0)$ to another involution in S_0. Thus $N_G(S_0)/C_G(S_0)$ must not be a two group and must also have order divisible by two since $S_0 \lhd G_2$. So $N_G(S_0)/C_G(S_0) \cong \Sigma_3$, and we notice this symmetric group permutes the elements x, y, and xy; furthermore, the character χ of G_2 given in Lemma 40 is again a 2-effective character in this case.

Case 4 was treated in Lemma 38. □

To conclude Section 5, we state the following theorem whose proof has now been completed.

Theorem 42. *For any rank two finite group G, G has a 2-effective character.*

6. Counterexample

Looking at the last section, one may wonder if a similar argument can be made for odd primes; for example could a p-effective character be found for any group involving Qd(p) and having the same Sylow p-subgroup as Qd(p). However, this is not the case as we will show in the following proposition.

Proposition 43. *Let p be an odd prime and G be a rank two finite group. If Qd(p) is involved in G and G and Qd(p) have isomorphic sylow p-subgroups, then G does not have a p-effective character.*

Proof. Let $G_p \in \mathrm{Syl}_p(G)$. Notice that G_p, which is isomorphic to a Sylow p-subgroup of Qd(p), is an extra-special p-group of size p^3 and exponent p. Let $\chi_1, \ldots, \chi_n$ be the irreducible characters of G_p and assume that χ_1 is the trivial character. Notice from the structure of G_p that if $\chi_i(1) \neq 1$, then for $g \in G_p$, $\chi_i(g) \neq 0$ if and only if $g \in Z(G_p)$. Also notice that $Z(G_p)$ is cyclic of order p and $Z(G_p)$ is not strongly closed in G_p with respect to G. Let $x \in Z(G_p)$ and $y \in G_p \setminus Z(G_p)$ such that there is a $g \in G$ with $y^g = x$. Suppose χ is a character of G_p that is a p-effective character of G. There exists $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$ such that $\chi = \sum_{i=1}^n a_i \chi_i$. Since χ is a p-effective character of G, it respects fusion in G, which implies $\chi(x) = \chi(y)$ and $a_1 = 0$. Suppose that $a_i > 0$ for some $i > 1$ with $\chi_i(1) = 1$. Fixing this i,

let $E_i = \{g \in G_p | \chi_i(g) = 1\}$. Notice that for such an i, E_i is a rank two elementary abelian subgroup of G_p. This implies that $[\chi_i|_{E_i}, 1_{E_i}] = 1$, so $[\chi|_{E_i}, 1_{E_i}] > 0$, which contradicts the assumption that χ is a p-effective character of G; therefore, for each i such that $\chi_i(1) = 1$, $a_i = 0$. Now

$$\chi(x) = \sum_{i \text{ such that } \chi_i(1) \neq 1} a_i \chi_i(x) + \sum_{i \text{ such that } \chi_i(1)=1} a_i$$

$$= \sum_{i \text{ such that } \chi_i(1)=1} a_i \chi_i(y) = \chi(y).$$

So $\chi(y) = 0$; therefore, $\chi(x) = 0$. Since $Z(G_p)$ is a cyclic group of order p, $\chi(z) = 0$ for all $z \in Z(G_p) \setminus \{1\}$. Notice that in showing $\chi(z) = 0$ for each $z \in Z(G_p) \setminus \{1\}$, we know that $\chi(g) = 0$ for all $g \in G_p \setminus Z(G_p)$ by the structure of G; thus, $\chi(g) = 0$ for all $g \in G_p \setminus \{1\}$. So we see that χ must be identically zero, which contradicts the definition of p-effective character. □

Lemma 44. *Let G be a finite group with p a prime dividing $|G|$ and $H \subseteq G$. Suppose that p divides $|H|$ and that $\mathrm{rk}_p(G) = \mathrm{rk}_p(H)$. If G has a p-effective character, so does H.*

Proof. We may assume that $\mathrm{rk}_p(H) = \mathrm{rk}(G)$; otherwise the result is obvious. Let $G_p \in \mathrm{Syl}_p(G)$ such that $H_p = G_p \cap H \in \mathrm{Syl}_p(H)$. Let χ be a character of G_p that is a p-effective character of G. Obviously $\chi|_{H_p}$ is a character of H_p, which is not identically zero. Since χ respects fusion in G, $\chi|_{H_p}$ respects fusion in H. Any maximal rank elementary abelian subgroup of H_p is also a maximal rank elementary abelian subgroup of G_p. The lemma follows easily from this fact. □

Combining Lemma 44 with Theorems 34 and 42, we get the following theorem from which Remark 21 follows:

Theorem 45. *Let G be a finite group of rank two and let p be a prime dividing $|G|$. G has p-effective character if and only if either $p = 2$ or both $p > 2$ and G does not p'-involve $Qd(p)$.*

In showing that a finite group G of rank two acts freely on a finite complex $Y \simeq \mathbb{S}^n \times \mathbb{S}^m$, we actually showed that G acts on a finite complex $X \simeq \mathbb{S}^m$ with isotropy groups of rank one and then applied Theorem 5. Ozgun Unlu [29] and Grodal [18] have each shown that for each odd prime p, $\mathrm{Qd}(p)$ cannot act on any finite complex homotopy equivalent to a sphere with rank one isotropy groups. From both proofs it is clear that for any odd prime p and rank two group G which p'-involves $\mathrm{Qd}(p)$, G cannot act on any finite complex homotopy equivalent to a sphere with rank one isotropy groups.

Combining this discussion with Theorem 45 and the discussion in Section 2, we conclude with the following proposition:

Proposition 46. *Let G be a finite group of rank two. G acts on some finite complex homotopy equivalent to a sphere with rank one isotropy groups if and only if for each prime odd prime p, G does not p'-involve $Qd(p)$.*

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