THE STRONG SYMMETRIC GENUS OF SMALL *D*- TYPE GENERALIZED SYMMETRIC GROUPS

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Michelle Bowser, Trevor Partridge and Kirsten Rodgers
Mathematics Department, Grove City College
Advisor: Michael A. Jackson

Notation

To describe a matrix, $S \in \sum_n$, we will use the notation $(1\ 2\ 3)$, which represents the matrix:

$$(1\ 2\ 3) \rightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Using this cycle notation, the order of the numbers says that row 1 goes to row 2, 2 goes to 3, and 3 goes to 1.

In addition we will describe the diagonal matrix, D, as $(0, 2, 1)$ to describe the $powers, k \pmod m$, on α as the appear in their $column$ order. In this case, the entries of the diagonal matrix are $0, \alpha^2, \alpha^1$. Thus $(0, 2, 1)$ represents the matrix:

$$[(0,2,1)] \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha^2 & 0 \\ 0 & 0 & \alpha \end{bmatrix}$$

The matrix $A \in D(n, m)$ can be written as the product SD as described above and will be represented by $[(1\ 2\ 3)\ (0, 2, 1)]$. Thus,

$$[(1\ 2\ 3)\ (0,2,1)] \rightarrow \begin{bmatrix} 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & \alpha^2 & 0 \end{bmatrix}$$

Approach to Solving the Problem

Basic Method

1. We started by using the computer program, Magma [1], to find the orders of the generators that produce the smallest genus.

2. Next, we worked by hand to find generators of the correct orders.

3. Lastly, we had to prove that any other orders that could produce a smaller genus were not actually generators.

Conjugation into a Maximal Subgroup

In order to conjugation into a maximal subgroup we used the following method:

1. We first chose two general elements, A and B, contained in D with the orders we needed to prove could not generate.

2. Then, we needed to show that there was another element, x, contained in G so that xAx^{-1} and xBx^{-1} had a list where all the elements were congruent to $0 \pmod m$.

If this is true, than you know that there is no way those elements could generate the whole group.

Sample Conjugation Proof

Lemma 1 *In $D(3, m)$, a triple of $2, 2, 3k$ cannot generate.*

Proof.
Our chosen elements are as follows:
$A : [(12), (a, -a, 0)](order2)$
$B : [(13), (b, 0, -b)](order2)$

Let $x : [(), (q, r, s)]$

Then:
$xAx^{-1} : [(12), (r+a-q, q-a-r, s+0-s)]$
$xBx^{-1} : [(13), (s+b-q, r-r, q-b-s)]$

Thus we need to find values for a, b and c such that
$r - q + a \equiv 0 (modm)$
$s + b - q \equiv 0 (modm)$

Therefore our two equations are:
$r - q + a \equiv 0 (modm)$
$s + b - q \equiv 0 (modm)$

so: pick any q and
$r \equiv q - a$
$s \equiv q - b$

Since this system of equations always has a solution, an x exists.

Basic Outline of the Problem

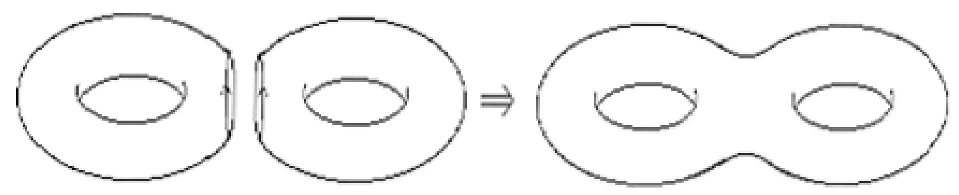
We found the Strong Symmetric Genus of Small D-Type Generalized Symmetric Groups of $D(n, m)$ with n either 3, 4, or 5.

Background of the Problem

To begin, think of a surface where the inside and the outside are distinct. We will call this surface **orientable**. For any surface where this is true, you can mold it (without cutting or tearing it) into a number of tori that are basically 'pasted' together.

Note that a **torus** is a shape that is similar to that of a donut.

For this problem we will be discussing the **genus** of a surface, which is simply the number of tori that are pasted together to make that surface.



From GAP-system Website

Note that every group **acts** on a surface, which means that it actually does something to the surface, such as flip or rotate it.

Our goal was to find the **Strong Symmetric Genus** of a group D, which is simply the smallest number of tori making up a surface that the group can still act on without changing it's orientation.[2]

Riemann-Hurwitz Equation

Riemann and Hurwitz developed an equation that finds the genus created by a group by using the order of the group and the order of the generators of the group.

Note that the **order of a group**, $|G|$, is the number of elements contained in a group, the **order of an element** is the number of times you multiply it by itself before you get the identity, and **generators** are elements in a group that when multiplied together in various patterns, end up coming up with every element in the group.

Suppose there exists two elements of D, A and B, that generate D. If A has order p, B has order q, and AB has order r, then there exists some surface on which D does something to the surface without changing it's orientation.

$$g = 1 + \frac{|G|}{2}\left(1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}\right).$$

Thus, in order to find the **smallest** genus, we wish to find the generating triple $\{A, B, AB\}$ that gives us the **largest** $\frac{1}{p} + \frac{1}{q} + \frac{1}{r}$. [3]

We refer to a generating triple $\{A, B, AB\}$ using its corresponding triple (p, q, r).

D-Type Generalized Symmetric Group

The **D-type generalized symmetric group**, $D(n, m)$, is a supbgroup of a group called the **generalized symmetric group**, $G(n, m)$. $G(n, m)$ is formed by allowing all the non-zero entries of an $n \times n$ permutation matrix to be elements with powers that are congruent to $0 \pmod m$ (also called m^{th} **roots of unity**).

The subgroup, $D(n, m)$, has all the same properties as $G(n, m)$, except the product of the non-zero entries has to be the identity, which means that the sum of the powers add up to $0 \pmod m$ every element we work with can only have a diagonal with powers that add up to $0 \pmod m$.

Results

Triples giving the Strong Symmetric Genus

$\boldsymbol{D(3,m)}$:	$(2,3,2m)$ when $3 \nmid m$
	$(2,2,3,3)$ when $D(3,m), 3 \vert m$
$\boldsymbol{D(4,m)}$:	$(2,4,9)$ for $m = 3$
	$(3,4,4)$ when $m > 3$
$\boldsymbol{D(5,m)}$:	$(3,4,4)$ when $m = 3$
	$(2,5,12)$ when $m = 6$
	$(2,4,6m)$ when , $m \neq 3$ or 6

The Strong Symmetric Genus

$\boldsymbol{D(3,m)}$:	$1 + \frac{m(m-3)}{2}$ when $3 \nmid m$
	$1 + m^2$ when $3 \vert m$
$\boldsymbol{D(4,m)}$:	46 for $m = 3$
	$1 + 2m^3$ when $m > 3$
$\boldsymbol{D(5,m)}$:	811 when $m = 3$
	16849 when $m = 6$
	$1 + 10m^3(3m-2)$ when , $m \neq 3$ or 6

Student Opportunities in Accelerated Research 2009 Team

Trevor Partridge, Kirsten Rodgers, Michelle Bowser

References

[1] W. Bosma, J. Cannon, and Ca. Playoust. *The Magma algebra system. I. The user language.* J. Symbolic Comput. **24** (1997), 235-265.

[2] J. L. Gross and T. W. Tucker. *Topological graph theory*, Wiley-Interscience, 1987.

[3] D. Singerman. *Symmetries of Riemann surfaces with large automorphism group.* Math. Ann. **210** (1974), 17-32.

This project was funded by the Swezey Fund for Scientific Research and Instrumentation.