A QUOTIENT OF THE SET $[BG, BU(n)]$ FOR A FINITE GROUP G OF SMALL RANK

MICHAEL A. JACKSON

Abstract. Let G_p be a Sylow p-subgroup of the finite group G and let $\mathrm{Char}_n^G(G_p)$ represent the set of degree n complex characters of G_p that are the restrictions of class functions on G. We construct a natural map $\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p)$ and prove that ψ_G is a surjection for all finite groups G that do not contain a subgroup isomorphic to $(\mathbb{Z}/p)^3$ for any prime p. We show, furthermore, that ψ_G is in fact a bijection for two types of finite groups G: those with periodic cohomology and those of odd order that do not contain a subgroup isomorphic to $(\mathbb{Z}/p)^3$ for any prime p.

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1. Introduction

Our purpose is to investigate the relationship between the homotopy classes of maps from BG for a finite group G to $BU(n)$ and the degree n characters of the Sylow p-subgroups of G. Throughout this paper we will let G be a finite group, p a prime dividing the order of G, and G_p a Sylow p-subgroup of G. In addition we will use $\mathrm{Char}_n(G_p)$ to represent the set of degree n complex characters of G_p and will let $\mathrm{Char}_n^G(G_p)$ represent the subset consisting of those degree n complex characters of G_p that are the restrictions of class functions on G.

Recall that for a group G, BG is the classifying space of G. Also $[BG, BU(n)]$ is the homotopy classes of maps from BG to $BU(n)$, which may also be thought of as $\pi_0\mathrm{Hom}(BG, BU(n))$ (see [12]). Next we will define a natural map

$$\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p),$$

which will be discussed throughout this paper. To define this map, let us examine the following diagram where $\mathrm{Rep}(G, U(n)) = \mathrm{Hom}(G, U(n))/\mathrm{Inn}(U(n))$:

$$\begin{array}{ccccc}
[BG, BU(n)] & \xrightarrow{\cong} & \prod_{p||G|}[BG, BU(n)^\wedge_p] & \xrightarrow{\mathrm{res}} & \prod_{p||G|}[BG_p, BU(n)^\wedge_p] \\
\bar{\psi}_G \downarrow & & \bar{\phi}_G \downarrow & & \cong \uparrow \\
\prod_{p||G|} \mathrm{Char}_n(G_p) & = & \prod_{p||G|} \mathrm{Char}_n(G_p) & \xleftarrow{\cong} & \prod_{p||G|} \mathrm{Rep}(G_p, U(n)).
\end{array}$$

Notice that spaces in the center and right of the top row contain $BU(n)^\wedge_p$, which is the p-completion of the space $BU(n)$. (For more information on p-completion, see [7, Chap. VI].) The bijection in the upper left is a result of work by Jackowski, McClure, and Oliver [16],

while the bijection on the far right is a result of work by Dwyer and Zabrodsky [12] and the bijection in the lower right is a basic result in representation theory. The restriction map res is induced by the inclusion of the Sylow p-subgroups G_p into G. We now let maps $\bar{\psi}_G$ and $\bar{\phi}_G$ be the maps that make the diagram commute. The image of $\bar{\psi}_G$ and $\bar{\phi}_G$ both lie in the subset $\prod_{p||G|} \mathrm{Char}_n^G(G_p) \subseteq \prod_{p||G|} \mathrm{Char}_n(G_p)$ (see Theorem 2.9). So we will let ψ_G and ϕ_G be the maps $\bar{\psi}_G$ and $\bar{\phi}_G$ respectively, with the range restricted to $\prod_{p||G|} \mathrm{Char}_n^G(G_p)$. Now that we have defined the map ψ_G, we state the following three theorems, which express the main results of this paper:

Theorem 1.1. *If G is a finite group that does not contain a rank two elementary abelian subgroup, then the natural mapping*

$$\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p)$$

is a bijection.

Theorem 1.2. *Let G be a finite group that does not contain a rank three elementary abelian subgroup. If $|G|$ is odd, then the natural mapping*

$$\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p)$$

is a bijection.

Theorem 1.3. *If G is a finite group that does not contain a rank three elementary abelian subgroup, then the natural mapping*

$$\psi_G : [BG, BU(n)] \to \prod_{p||G|} \mathrm{Char}_n^G(G_p)$$

is a surjection.

The results in this paper build on work by Mislin and Thomas [20], who a prove similar result to Theorem 1.1 where $U(n)$ is replaced by $SU(2)$. This work is also related to work by Jackowski and Oliver [17]. They look at the Grothendieck group of $\mathrm{Vect}(BG)$ and show that it is isomorphic to $\prod_{p||G|} \mathcal{R}(G_p)^G$, where $\mathcal{R}(G_p)$ is the complex representation ring of G_p restricted to thee elements that are stable under the action of G. The present work is intended to be a step toward classifying finite groups that act freely on a finite CW-complex that is homotopy equivalent to the product of two spheres. The classification of such groups began with Adem and Smith [1, 2]. Using Theorem 1.3 and this author's thesis [18], a finite group can be shown to act freely on a finite CW-complex that is homotopy equivalent to the product of two spheres by demonstrating the appropriate element of $\prod_{p||G|} \mathrm{Char}_n^G(G_p)$. Such an element must be in the product of characters, not virtual characters, which correspond

to the representation ring as used by Jackowski and Oliver. A more complete explanation of this application of the present work will be explored in a subsequent paper.

2. A homology decomposition and conjugation families

We start by showing that we can work with each prime dividing the order of G separately. Notice that for the diagram from Section 1, the maps, excluding maps from $[BG, BU(n)]$, can each be separated into a product over all $p||G|$, yielding a separate diagram for each $p||G|$:

$$\begin{array}{ccc} [BG, BU(n)^\wedge_p] & \xrightarrow{\mathrm{res}_p} & [BG_p, BU(n)^\wedge_p] \\ \bar{\phi}_{G,p}\Big\downarrow & & \cong\Big\uparrow \\ \mathrm{Char}_n(G_p) & \xleftarrow{\cong} & \mathrm{Rep}(G_p, U(n)). \end{array}$$

Notice that the image of the map $\bar{\phi}_{G,p}$ is always contained in the subset $\mathrm{Char}_n^G(G_p) \subseteq \mathrm{Char}_n(G_p)$. So we will let $\phi_{G,p}$ be the map $\bar{\phi}_{G,p}$ with the range restricted to $\mathrm{Char}_n^G(G_p)$. We will show the image under certain hypotheses is all of $\mathrm{Char}_n^G(G_p)$ by looking at the map res_p and by introducing a homology decomposition of BG.

We will use the subgroup decomposition as our homology decomposition, which can be described in the following manner. First let $\mathcal{C}$ be a collection of p-subgroups of G closed under conjugation. Let $\mathcal{O}_\mathcal{C}$ be the $\mathcal{C}$-orbit category, which is the category with objects G/P for $P \in \mathcal{C}$ and with G-maps as the morphisms. Let $\mathcal{I}$ be the inclusion functor from $\mathcal{O}_\mathcal{C}$ to the category of G-spaces. Composing $\mathcal{I}$ with the Borel construction $(-)_{hG}$ gives a functor $\alpha_\mathcal{C} : \mathcal{O}_\mathcal{C} \to$ **Spaces**. Notice that $\alpha_\mathcal{C}(G/P)$ has the homotopy type of BP for any p-subgroup $P \subseteq G$. This functor naturally induces another functor $b_\mathcal{C} : \mathrm{hocolim}\ \alpha_\mathcal{C} \to BG$; therefore, $b_\mathcal{C}$ gives a homology decomposition of BG if and only if $\mathcal{C}$ is an ample collection of subgroups of G (see [11]).

At this point an ample collection of subgroups of G will be discussed, starting with three definitions.

Definition 2.1. Let $P \subseteq G$ be a p-subgroup. P is said to be *p-radical* (or p-stubborn) if $N_G(P)/P$ has no non-trivial normal p-subgroups.

Definition 2.2. Let $P \subseteq G$ be a p-subgroup. P is said to be *p-centric* if $Z(P)$ is a Sylow p-subgroup of $C_G(P)$.

Definition 2.3 (see [14])**.** Let $P \subseteq G$ be a p-centric subgroup. P is said to be *principal p-radical* if $N_G(P)/PC_G(P)$ has no non-trivial normal p-subgroups.

Now let the collection $\mathcal{C}$ be the set of all principal p-radical subgroups of G. Grodal has shown that this is an ample collection [14], allowing for a homology decomposition of G. Also it should be noted that any Sylow p-subgroup of G is contained in the collection $\mathcal{C}$ and that any principal p-radical subgroup of G is necessarily p-radical.

Next notice that the map res_p factors through an inverse limit constructed via the homology decomposition described above:

$$res_p : [BG, BU(n)^\wedge_p] \xrightarrow{\alpha_p} \lim_{\leftarrow G/P \in \mathcal{O}_\mathcal{C}} [BP, BU(n)^\wedge_p] \xrightarrow{\beta_p} [BG_p, BU(n)^\wedge_p].$$

The map α_p is induced by restriction and the map β_p is a projection onto a particular element since $G_p \in \mathcal{C}$. The following diagram then commutes where the maps π^1_p and π^2_p are projection onto the the set of degree n characters of G_p and onto the set of representations of G_p respectively:

$$\begin{tikzcd}
{[BG, BU(n)^\wedge_p]} \arrow[r, equal] \arrow[d, "\alpha_p"'] & {[BG, BU(n)^\wedge_p]} \arrow[d, "\mathrm{res}_p"'] \\
\lim\limits_{\leftarrow G/P \in \mathcal{O}_\mathcal{C}} [BP, BU(n)^\wedge_p] \arrow[r, "\beta_p"] & {[BG_p, BU(n)^\wedge_p]} \\
\lim\limits_{\leftarrow G/P \in \mathcal{O}_\mathcal{C}} \mathrm{Rep}(P, U(n)) \arrow[r, "\pi^2_p"] \arrow[u, "\cong"] \arrow[d, "\cong"'] & \mathrm{Rep}(G_p, U(n)) \arrow[u, "\cong"] \arrow[d, "\cong"'] \\
\lim\limits_{\leftarrow G/P \in \mathcal{O}_\mathcal{C}} \mathrm{Char}_n(P) \arrow[r, "\pi^1_p"] & \mathrm{Char}_n(G_p).
\end{tikzcd}$$

The bijections on the left side follow easily from the earlier discussion of p-groups. We notice that the composite of the entire right hand column is the map $\bar{\phi}_{G,p}$ whose image we want to find. From the diagram it is obvious that we can instead find the image of the map π^1_p.

In order to look closer at the image of the map π^1_p, Alperin's fusion theory will be discussed. In particular the definition of weak conjugation family and an example will be given.

Definition 2.4 (Alperin [3])**.** Let G be a finite group, p a prime dividing $|G|$, and G_p a Sylow p-subgroup of G. A set $\mathcal{F}$ of pairs $\{(H, T)\}$, where $H \subseteq G_p$ and $T \subseteq N_G(H)$, is called a *weak conjugation family* provided that whenever A and B are subsets of G_p and $B = g^{-1}Ag$ for $g \in G$, there are elements $(H_1, T_1), (H_2, T_2), \ldots, (H_n, T_n)$ of $\mathcal{F}$ and elements $x_1, x_2, \ldots, x_n, y$ of G such that

(1) $B = (x_1x_2 \cdots x_ny)^{-1}A(x_1x_2 \cdots x_ny)$,
(2) $x_i \in T_i$ for $1 \leq i \leq n$ and $y \in N_G(G_p)$, and
(3) $A \subseteq H_1$, $(x_1x_2 \cdots x_i)^{-1}A(x_1x_2 \cdots x_i) \subseteq H_{i+1}$ for $1 \leq i \leq n-1$.

The example we will be using of a weak conjugation family is given in the following theorem of Goldschmidt. In order to state the theorem we need to give two more definitions.

Definition 2.5 (Alperin [3])**.** Let Q and R be Sylow p-subgroups of a finite group G and let $H = Q \cap R$. H is said to be a *tame intersection* if $N_Q(H)$ and $N_P(H)$ are Sylow p-subgroups of $N_G(H)$.

Definition 2.6. A finite group G is called *p-isolated* if it contains a proper subgroup $H \subset G$ such that if $p||H|$ and for any $g \in G \setminus H$, $p \not| |H \cap gHg^{-1}|$. In this case H is called a strongly p-embedded subgroup of G.

Theorem 2.7 (Goldschmidt [13, Theorem 3.4])**.** *Let G be a finite group, p a prime dividing $|G|$, and G_p a Sylow p-subgroup of G. Let $\mathcal{F}_0$ be the set of all pairs $(H, N_G(H))$ where $H \subseteq G_p$ such that there exists a Sylow p-subgroup P of G with the following properties:*

1. *$H = G_p \cap P$ a tame intersection,*
2. *$C_{G_p}(H) \subseteq H$,*
3. *H a Sylow p-subgroup of $O_{p',p}(N_G(H))$, and*
4. *$H = P$ or $N_G(H)/H$ is p-isolated.*

$\mathcal{F}_0$ is a weak conjugation family.

The next proposition will begin to relate Alperin's fusion theory with the inverse limit that has been used in this paper. This will allow us to find the image of the map π_p^1.

Proposition 2.8. *Let G be a finite group, p a prime dividing $|G|$, and G_p a Sylow p-subgroup of G. The set $\mathcal{F}$ consisting of all pairs $(H, N_G(H))$, where $H \subseteq G_p$ is a principal p-radical subgroup of G, is a weak conjugation family.*

Proof. We will show that $\mathcal{F}$ is a weak conjugation family by showing that as a set it contains $\mathcal{F}_0$, the weak conjugation family of Goldschmidt. Let $(H, N_G(H)) \in \mathcal{F}_0$. We first notice that H is a p-centric subgroup of G since $C_{G_p}(H)$ is a Sylow p-subgroup of $C_G(H)$ by the fact that H is a tame intersection. Since H is a Sylow p-subgroup of $O_{p',p}(N_G(H))$ and H is p-centric, H must also be principal p-radical (see [14, Remark 10.12]). □

Theorem 2.9. *Let G be a finite group, p a prime dividing $|G|$, and G_p a Sylow p-subgroup of G. If $\mathcal{C}$ is the collection of all p-subgroups $H \subseteq G$ that are prinicipal p-radical, then the projection map*

$$\pi_p^1 : \lim_{\leftarrow G/P \in \mathcal{O}_{\mathcal{C}}} \mathrm{Char}_n(P) \to \mathrm{Char}_n(G_p)$$

is one to one and is onto the subset $\mathrm{Char}_n^G(G_p) \subseteq \mathrm{Char}_n(G_p)$.

Proof. First we will show that the image of π_p^1 is contained in $\mathrm{Char}_n^G(G_p)$. Suppose that $\gamma \in \lim_{\leftarrow G/P \in \mathcal{O}_{\mathcal{C}}} \mathrm{Char}_n(P)$. Define a map π_P for each $P \in \mathcal{C}$ as the projection from the inverse limit of the complex character of P. Notice that $\pi_p^1 = \pi_{G_p}$. Next define $\chi_P = \pi_P(\gamma)$ and let χ mean χ_{G_p}. Let $\mathcal{F}_p$ be the set consisting of all pairs $(H, N_G(H))$ where $H \in \mathcal{C}$ and $H \subseteq G_p$. By the last proposition, $\mathcal{F}_p$ is a weak conjugation family.

Suppose that $a, b \in G_p$ such that $\exists\ g \in G$ with $b = gag^{-1}$. By the definition of weak conjugation family, there exists $H_1, H_2, \ldots, H_m \in \mathcal{C}$ with $H_i \subseteq G_p$ for $1 \leq i \leq n$, and $x_1, x_2, \ldots x_m, y \in G$ such that

- $b = (x_1x_2\cdots x_ny)^{-1}a(x_1x_2\cdots x_ny)$,
- $x_i \in N_G(H_i)$ for $1 \leq i \leq n$,
- $y \in N_G(G_p)$, and
- $a \subseteq H_1$, $(x_1x_2\cdots x_i)^{-1}a(x_1x_2\cdots x_i) \subseteq H_{i+1}$ for $1 \leq i \leq n-1$.

For each $1 \leq i \leq n$ we notice that $\chi|_{H_i}$ respects fusion in $N_G(H_i)$ and χ respects fusion in $N_G(G_p)$. From this we see that the following three statements hold:

- $\chi(a) = \chi(x_1^{-1}ax_1)$,
- $\chi((x_1x_2\cdots x_i)^{-1}a(x_1x_2\cdots x_i)) = \chi((x_1x_2\cdots x_{i+1})^{-1}a(x_1x_2\cdots x_{i+1}))$ for each $1 \leq i \leq n-1$, and
- $\chi((x_1x_2\cdots x_n)^{-1}a(x_1x_2\cdots x_n)) = \chi(b)$.

Putting these statements together, we see that $\chi(a) = \chi(b)$; therefore, χ respects fusion in G and so is contained in $\mathrm{Char}_n^G(G_p)$.

To show that π_p^1 is a one to one correspondence, it is enough to show the existence of an inverse mapping

$$\sigma_{G_p} : \mathrm{Char}_n^G(G_p) \to \lim_{\leftarrow G/P \in \mathcal{O}_{\mathcal{C}}} \mathrm{Char}_n(P).$$

Let $\chi \in \mathrm{Char}_n^G(G_p)$. We will define σ_{G_p} by giving the characters χ_P, which will be $\pi_P \circ \sigma_{G_p}(\chi)$ for each $P \in \mathcal{C}$. Fix $P \in \mathcal{C}$. There exists $g \in G$ such that $P \subseteq g^{-1}G_pg$. So let $\chi_P(h) = \chi(ghg^{-1})$. Since χ respects fusion in G, it is easy to see that this definition of $\chi_P = \pi_P \circ \sigma_{G_p}(\chi)$ is well defined. This definition then gives the following definition of σ_{G_p}:

$$\sigma_{G_p}(\chi) = \lim_{\leftarrow G/P \in \mathcal{O}_{\mathcal{C}}} \chi_P.$$

It is obvious from the definition that $\pi_p^1 \circ \sigma_{G_p}$ is the identity mapping. We now have to show only that $\sigma_{G_p} \circ \pi_p^1$ is the identity mapping. Let γ be an element of the inverse limit and let $\chi = \pi_{G_p}(\gamma)$. Given $P \in \mathcal{C}$, observe that $\pi_P(\gamma)$ must be χ_P defined above by the nature of the $\mathcal{C}$-orbit category $\mathcal{O}_{\mathcal{C}}$. This observation shows that $\pi_P \circ \sigma_{G_p} \circ \pi_p^1 = \pi_P$ for each $P \in \mathcal{C}$; therefore, $\sigma_{G_p} \circ \pi_p^1$ must be the identity mapping. $\square$

Applying Theorem 2.9 shows that the map π_p^1 is a bijection when the range is restricted to $\mathrm{Char}_n^G(G_p)$. Using this result we see that we can use the maps $\phi_{G,p}$, ϕ_G, and ψ_G instead of the corresponding maps $\bar{\phi}_{G,p}$, $\bar{\phi}_G$, and $\bar{\psi}_G$. We also see that if the map α_p is an injection or a surjection, so is the map $\phi_{G,p}$. In order to study the map α_p, we must apply obstruction theory.

3. Obstruction theory

Recall the functor $\alpha_{\mathcal{C}} : \mathcal{O}_{\mathcal{C}} \to \mathbf{Spaces}$, which we encountered in Section 2. Fixing an element

$$\gamma = (\gamma_{(G/P)})_{G/P \in \mathcal{O}_{\mathcal{C}}} \in \lim_{\leftarrow G/P \in \mathcal{O}_{\mathcal{C}}} [\alpha_{\mathcal{C}}(G/P), BU(n)^{\wedge}_p],$$

we define functors $\mathcal{D}_n : \mathcal{O}_{\mathcal{C}} \to \mathcal{A}b$ for each $n \geq 1$ by letting

$$\mathcal{D}_n(G/P) = \pi_n(\mathrm{map}(\alpha_{\mathcal{C}}(G/P), BU(n)^{\wedge}_p)_{\gamma_{(G/P)}}).$$

At this point in our discussion we will use a theorem by Jackowski, McClure, and Oliver [16]. This particular theorem explains where the obstructions will lie in our examination of the map α_p.

Theorem 3.1 (Jackowski, McClure, and Oliver [16])**.** *Fix an element* $\gamma \in \lim_{\leftarrow G/P \in \mathcal{O}_{\mathcal{C}}} [\alpha_{\mathcal{C}}(G/P), BU(n)^{\wedge}_p]$. $\gamma \in \mathrm{Im}(\alpha_p)$ *if the groups* $\lim^{n+1}(\mathcal{D}_n)$ *vanish for all* $n \geq 1$ *and* $\alpha_p^{-1}(\gamma)$ *contains at most one element if the groups* $\lim^{n}(\mathcal{D}_n)$ *vanish for all* $n \geq 1$.

Since we will are trying to show that α_p is an injection or a surjection, we will use the following corollary to Theorem 3.1:

Corollary 3.2. *If for each element* $\gamma \in \lim_{\leftarrow_{G/P\in\mathcal{O}_\mathcal{C}}}[\alpha_\mathcal{C}(G/P), BU(n)^\wedge_p]$, $\lim^{n+1}(\mathcal{D}_n)$ *vanish for all* $n \geq 1$*, then* α_p *is a surjection. On the other hand, if for each* γ, $\lim^n(\mathcal{D}_n)$ *vanish for all* $n \geq 1$*, then* α_p *is an injection.*

For additional discussion of the obstruction theory see [6, 7, 14, 16, 24]. In light of Corollary 3.2, we are left to show that under the correct hypotheses the groups $\lim^{n+1}(\mathcal{D}_n)$ and $\lim^n(\mathcal{D}_n)$ vanish for $n \geq 1$. We will notice first that in the case where $n = 1$, these groups vanish without any additional hypotheses. Recall that for any representation $\rho : P \to U(m)$ with P a p-group,

$$\pi_n(\mathcal{D}_\mathcal{C}(G/P), B\rho) \cong \pi_n(BC_{U(m)}(Im(\rho))) \otimes \mathbb{Z}^\wedge_p \text{ (see [12])}.$$

Since the centralizer of a finite p-subgroup of $U(m)$ is the product of various $U(i)$, it is clear that $BC_{U(m)}(Im(\rho))$ is simply connected; therefore, for any element γ, $\lim^2(\mathcal{D}_1) = 0 = \lim^1(\mathcal{D}_1)$. In the remaining sections we will discuss situations where the remaining higher limits are indeed trivial.

4. Vanishing obstruction groups

Before we proceed in showing when these higher limits vanish, we first need a couple of definitions. These will allow us to use standard theory first introduced by Jackowski, McClure, and Oliver [15, 16].

Definition 4.1. A functor $F : \mathcal{O}_\mathcal{C} \to Ab$ is called an *atomic* functor if it vanishes on all but possibly a single isomorphism class of objects. An atomic functor will be said to be concentrated on an isomorphism class on which it does not vanish.

Definition 4.2 (Definition 4.7 of [16])**.** For any prime p, finite group G, and $\mathbb{Z}_{(p)}(G)$-module M, let $F_M : \mathcal{O}_{\mathcal{S}_p(G)} \to \mathbb{Z}_{(p)} - mod$ be the atomic functor concentrated on the free orbit $G/1$ with $F_M(G/1) = M$. Set $\Lambda^*(G; M) = \lim^* F_M$.

The following lemma follows directly from work by Jackowski, McClure, and Oliver, in particular from the first proof of Lemma 5.4 in [15].

Lemma 4.3. *Suppose* $\mathcal{C}$ *is a subcollection of* $\mathcal{S}_p(G)$ *for a finite group* G *such that for any functor* $F : \mathcal{O}_{\mathcal{S}_p(G)} \to \mathbb{Z}_{(p)} - mod$, $\lim^* F \cong \lim^* F|_{\mathcal{O}_\mathcal{C}}$. *Also assume that* $F' : \mathcal{O}_\mathcal{C} \to \mathbb{Z}_{(p)} - mod$ *is an atomic functor concentrated on the isomorphism class of* G/P *for some* $P \in \mathcal{C}$. *Then*

$$\lim^* F' \cong \Lambda^*(N_G(P)/P; F'(G/P)).$$

We will be filtering our functor by atomic functors in order to see where $\Lambda^*(\Gamma, M)$ vanishes. The relevant lemma is a result of work by Grodal [14]. (See also [16, Prop. 4.11] and [8, Proposition 5.8].)

Lemma 4.4 (See [14])**.** *Let* Γ *be a finite group and* M *a finitely generated* $\mathbb{Z}_{(p)}(\Gamma)$*-module. If* m *is a non-negative integer such that* $\mathrm{rk}_p(\Gamma) \leq m$*, then* $\Lambda^j(\Gamma, M) = 0$ *for any* $j > m$. *In particular, if* $p^{m+1} \not| |\Gamma|$*, then* $\Lambda^j(\Gamma, M) = 0$ *for any* $j > m$.

Lemma 4.4 gives us the following proposition when a functor is filtered by atomic functors.

Proposition 4.5. *Let G be a finite group and p a prime number dividing $|G|$. Let $\mathcal{C}$ be the set of all subgroups of G that are principal p-radical. If m is a positive integer such that for each $P \in \mathcal{C}$ $\mathrm{rk}_p(N_G(P)/P) \leq m$, then for any functor $F : \mathcal{O}_{\mathcal{C}} \to \mathbb{Z}_{(p)} - mod$, $\lim^j F = 0$ for any $j > m$.*

Using this proposition in conjunction with the work of the previous section, we get the following result.

Theorem 4.6. *Let G be a finite group and p a prime number dividing $|G|$. Let $\mathcal{C}$ be the set of all subgroups of G that are principal p-radical. If for each $P \in \mathcal{C}$ $\mathrm{rk}_p(N_G(P)/P) \leq 2$, then the map α_p described earlier is a surjection. If for each $P \in \mathcal{C}$ $\mathrm{rk}_p(N_G(P)/P) \leq 1$, then the map α_p is a bijection.*

In the rest of this paper, we will use Theorem 4.6 and various results about finite groups to prove the three theorems presented in the introduction.

5. Finite groups, part I

We will start by proving Theorem 1.1. We begin this process with the following lemma, which shows that the obstructions vanish when a Sylow p-subgroup of G has a center of small index in itself.

Lemma 5.1. *Let G be a finite group, p a prime number dividing $|G|$, and G_p a Sylow p-subgroup of G. Let m be an integer such that $p^m = [G_p : Z(G_p)]$. If P is a p-centric subgroup of G, then $p^m \not| [N_G(P) : P]$. In particular, if G_p is abelian, then the only p-centric subgroups of G are the Sylow p-subgroups.*

Proof. Let $P \subseteq G$ be a p-centric subgroup, which is not a Sylow p-subgroup of G. We may assume $P \subset G_p$, which implies that $Z(G_p) \subseteq C_G(P) \cap G_p$. Let $H = C_G(P) \cap G_p$, which is a Sylow p-subgroup of $C_G(P)$. It is clear that $Z(P) \subseteq H$, which implies that $Z(P) = H$ since P is p-centric; therefore, $Z(G_p) \subseteq Z(P) \subseteq P$. Suppose that $Z(G_p) = P$, then $G_p \subseteq C_G(P)$, contradicting P being p-centric. So we see that $Z(G_p)$ is strictly contained in P. Now by hypothesis, $p^m = [G_p : Z(G_p)]$; thus, by the strict containment $Z(G_p) \subset P$, $[G_p : P] < p^m$. This clearly implies $p^m \not| [G : P]$. Then since $[N_G(P) : P]$ divides $[G : P]$, $p^m \not| [N_G(P) : P]$. □

At this point we will prove Theorem 1.1:

Proof. As was discussed earlier, it is enough to show that for each prime p that divides $|G|$, $\phi_{G,p}$ is a bijection. We assume that G does not contain a rank two elementary abelian subgroup. Let G_p be a Sylow p-subgroup of G. If G_p is cyclic, it is abelian; therefore, the only p-centric subgroups of G are the Sylow p-subgroups. By Theorem 4.6, the map α_p is a bijection as is the map $\phi_{G,p}$. On the other hand, if G_p is not a cyclic group, then $p = 2$

and G_p is a generalized quaternion group. Suppose in this case that $P \subseteq G_p$ is a p-centric subgroup of G. We see that either P is the cyclic group of index two in G_p, P is cyclic of order four and intersects the cyclic group of index two in G_p in a cyclic group of order two, or P itself is a generalized quaternion group. If P is of index two in G_p, $4 \nmid [N_G(P) : P]$. In the second case, $N_{G_p}(P)$ is a quaternion group of order eight and $4 \nmid [N_G(P) : P]$. If P is generalized quaternion and not a Sylow p-subgroup, it follows that $N_{G_p}(P)$ is a generalized quaternion group with $[N_{G_p}(P) : P] = 2$ while if P is a Sylow p-subgroup, $[N_{G_p}(P) : P] = 1$. We have now proven in each case that if P is a p-centric subgroup of G, then $p^2 \nmid [N_G(P) : P]$; furthermore, the map α_p is a bijection. From our previous discussion, we see that the map $\phi_{G,p}$ is also a bijection. □

The proof of Theorem 1.2 follows from the next lemma.

Lemma 5.2 (See [21, 6.3.10])**.** *Let G be a finite group of odd order. If $p||G|$ and P is a principal p-radical subgroup of G such that* $\mathrm{rk}(P) \leq 2$*, then P is a Sylow p-subgroup of G.*

Proof. Let $p||G|$ and P be a principal p-radical subgroup of G such that $\mathrm{rk}(P) \leq 2$. Suppose P is not a Sylow p-subgroup of G. Let $W = N_G(P)/C_G(P)$. The derived subgroup W' of W is a p-subgroup (see [21, 6.3.10]). For a subgroup H of $N_G(P)$, we will denote by $\bar{H}$ the image of $HC_G(P)$ in W. Now since P is principal p-radical, $O_p(W) = \bar{P}$. This implies $W' \subseteq \bar{P}$. So $W/\bar{P} = N_G(P)/PC_G(P)$ is an abelian group. Since P is not a Sylow p-subgroup of G, $p|[N_G(P) : P]$. P is principal p-radical; therefore, $Z(P)$ is a Sylow p-subgroup of $C_G(P)$. This shows that $p|[N_G(P) : PC_G(P)]$; thus, $N_G(P)/PC_G(P)$ has a non-trivial normal p-subgroup. This contradicts P being a principal p-radical subgroup of G, so P is a Sylow p-subgroup of G. □

According to Lemma 5.2, if G is a rank two group of odd order with $p||G|$, then the map α_p is a bijection, and the map $\phi_{G,p}$ is also a bijection. It follows from the introduction that if G is a rank two group of odd order, then the map ϕ_G is also a bijection, thus proving Theorem 1.2.

This only leaves Theorem 1.3 to be proven. As we have been doing in order to show that the map ϕ_G is a surjection, we will be showing that the map $\phi_{G,p}$ is a surjection for each prime p dividing $|G|$. We will need to approach the prime 2 differently then the odd primes. Thus, we will break Theorem 1.3 into the following two propositions.

Proposition 5.3. *Let G be a finite group and $p > 2$ an odd prime dividing $|G|$. If G does not contain a rank three elementary abelian p-subgroup, then $\phi_{G,p} : [BG, BU(n)^\wedge_p] \to \mathrm{Char}^G_n(G_p)$ is a surjection.*

Proposition 5.4. *If G be a finite group of even order not containing a rank three elementary abelian 2-subgroup, then $\phi_{G,2} : [BG, BU(n)^\wedge_2] \to \mathrm{Char}^G_n(G_2)$ is a surjection.*

It is clear that Theorem 1.3 follows immediately from these two propositions. We will spend the next two sections proving these two propositions.

6. 3-Groups of Maximal Class

We will prove Proposition 5.3 first. Before we do, we need to focus on a type of group that will arise in this proof. For Section 6 we will focus on the case where a Sylow 3-subgroup of G is a 3-group of maximal class with the purpose of proving the following lemma.

Lemma 6.1. *Let G be a finite group and G_3 be a Sylow 3-subgroup of G such that G_3 is a 3-group of maximal class and $|G_3| \geq 3^5$. If $P \subseteq G_3$ is a 3-centric subgroup of G, then $[N_{G_3}(P) : P] \leq 3^2$.*

Proof. We will let the series

$$Z(G_3) = C_{n-1} \subset C_n \subset \cdots \subset C_3 \subset C_2 \subset G_3$$

be the lower central series of G_3. This is defined inductively by $C_2 = [G_3, G_3]$ and $C_i = [C_{i-1}, G_3]$ for $i > 2$. Since G_3 is of maximal class, n is the integer such that $|G_3| = 3^n$. We also introduce another subgroup of G_p, which will be denoted by C_1. The subgroup C_1 is defined by the property that C_1/C_4 is the centralizer in G_3/C_4 of C_2/C_4 (see [5, section 2]). By these definitions we note that each C_i is characteristic in G_p, thus giving the increasing sequence of $n-1$ distinct proper subgroups of G_p:

$$Z(G_3) = C_{n-1} \subset C_n \subset \cdots \subset C_3 \subset C_2 \subset C_1 \subset G_3.$$

By the work of Blackburn [5], either C_1 is abelian or the commutator subgroup C_2 is abelian and $[C_1, C_2] = Z(G_3)$. Now assume that P is a proper subgroup of G_3 that is a 3-centric subgroup of G. We will look at two cases: the first when $P \subseteq C_1$ and the second when $P \not\subseteq C_1$.

Case I: Assume that $P \subseteq C_1$. Since P is a 3-centric subgroup of G, $Z(C_1)$ must be a proper subgroup of P. Notice that if either C_1 is abelian or the commutator subgroup C_2 is abelian and $[C_1, C_2] = Z(G_3)$, then $C_3 \subseteq Z(C_1)$. This inclusion implies that $[G_3 : P] \leq 3^2$.

Case II: Assume that $P \not\subseteq C_1$. Let $N = N_{G_3}(P)$. Let t be an element of $P \setminus (P \cap C_1)$. It can be shown that $|C_{G_3}(t)| = 3^2$ and that $C_{G_3}(t) \subseteq P$; therefore, any subgroup of G_3, which contains P, is of maximal class.

Suppose that P is abelian. This implies that $P = C_{G_3}(t)$ and that N is generated by t and C_{n-2}; thus $[N : P] = 3$.

Now suppose that P is not abelian. Since N is of maximal class, $[N, N] \subseteq C_2$; therefore, $[N, N]$ is abelian. Seeing that $P \lhd N$ and P is not abelian, $[N : P] = 3$. $\square$

7. Finite groups, part II

We need to mention one more lemma before we proceed with the proof of Proposition 5.3. This Lemma deals with the case where a Sylow p-subgroup of G is a metacyclic p-group and p is an odd prime. It follows from the work of J. Dietz [9] and of Martino and Priddy [19].

Lemma 7.1 (See [9], and [19, Theorem 2.7])**.** *Let G be a finite group, p an odd prime divisor of $|G|$, and G_p a Sylow p-subgroup of G. If G_p is a metacyclic group, $|G_p| > p^3$, and $P \subseteq G_p$ is a principal p-radical subgroup of G, then $P = G_p$.*

Proof. Suppose G_p is a metacyclic group with $|G_p| > p^3$. Martino and Priddy [19] have shown that such a G_p is a Swan Group. In the proof they proved that if $H \subset G_p$ is a proper subgroup with $C_{G_p}(H) = Z(H)$, then

$$(N_{G_p}(H)/HC_{G_p}(H)) \cap O_p(\mathrm{Out}(H)) \neq \{1\}.$$

So no proper subgroup $H \subset G_p$ can be a principal p-radical subgroup of G. □

Now we give the proof of Proposition 5.3:

Proof. Recall that G is a finite group that does not contain a rank three elementary abelian subgroup and p is an odd prime dividing $|G|$. Let G_p be a Sylow p-subgroup of G and let n be an integer such that $|G_p| = p^n$. Since Theorem 1.1 takes care of the case where $\mathrm{rk}(G_p) = 1$, we may assume that $\mathrm{rk}(G_p) = 2$. It has been shown by Blackburn [4] that one of the following holds for G_p (see also [10]):

(1) $n < 5$,
(2) $n \geq 5$ and G_p is metacylic,
(3) $n \geq 5$, $p = 3$, and G_3 is a 3-group of maximal class,
(4) $n \geq 5$ and $G_p = \langle a, b, c | a^p = b^p = c^{p^{n-2}} = 1, [a, b] = c^{p^{n-3}}, c \in Z(G_p)\rangle$, or
(5) $n \geq 5$, $e \neq 0$ is a quadratic nonresidue mod p, and
$G_p = \langle a, b, c | a^p = b^p = c^{p^{n-2}} = [b, c] = 1, [a, b^{-1}] = c^{ep^{n-3}}, [a, c] = b\rangle$.

Recall that it is enough to show that in each of these cases, if $P \subseteq G_p$ is a principal p-radical subgroup of G, then $[N_{G_p}(P) : P] \leq p^2$. Lemma 7.1 gives this result for case 2, while case 3 was shown in Section 6. To proceed with the other three cases, recall that by Lemma 5.1 it is enough to show that $[G_p, Z(G_p)] \leq p^3$. The first case follows immediately since G_p must have a non-trivial center. Notice in both of the remaining cases that the center of G_p contains the element c^p implying that $|Z(G_p)| \geq p^{n-3}$. This concludes these two cases as well as the proof of Proposition 5.3. □

We are also ready to give the proof of of Proposition 5.4:

Proof. Recall that G is a finite group of even order that does not contain a rank three elementary abelian subgroup. As before let G_2 be a Sylow 2-subgroup of G. Suppose that $P \subset G_2$ is a principal 2-radical subgroup of G, which is not itself a Sylow 2-subgroup of G. Notice that $N_G(P)/PC_G(P) \subseteq \mathrm{Out}(P)$ is both non-trivial and not a 2-group. In particular

this means that P is a 2-group of rank one or two with an automorphism of odd order. Richard Thomas has classified all such 2-groups [22, 23]. According to his classification P must be one of the following:

(1) $P \cong Q_8 * D_8$,
(2) $P \cong Q_8 \times Q_8$,
(3) $P \cong Q_8 \wr \mathbb{Z}_2$,
(4) $P \cong Q_8 * C$ where $\mathrm{Out}(C)$ is a 2-group, $\mathrm{Out}(C) \lhd \mathrm{Out}(P)$, and $C \neq D_8$,
(5) $P \cong \mathbb{Z}_{2^r} \times \mathbb{Z}_{2^r}$, or
(6) $P \cong U_{64} \in \mathrm{Syl}_2(\mathrm{PSU}_3(\mathbb{F}_4))$.

Our notation is as follows: Q_8 means the quaternion 8 group, D_8 means the dihedral group of size 8, $*$ refers to the central product, and $\wr$ refers to the wreath product.

Since P is a principal 2-radical subgroup of G, we know from the definition that $N_G(P)/PC_G(P) \cap O_2(\mathrm{Out}(P)) = \{1\}$; therefore, let $S \in \mathrm{Syl}_2(N_G(P)/PC_G(P))$ and $T \in \mathrm{Syl}_2(\mathrm{Out}(P)/O_2(\mathrm{Out}(P)))$. It is clear then that S must be isomorphic to a subgroup of T. In order to show that $\mathrm{rk}_2(N_G(P)/P) \leq 2$, it is enough to show that $\mathrm{rk}_2(T) \leq 2$. We will do this on a case by case basis using the list above. In cases 1 and 2, $T \cong D_8$, giving $\mathrm{rk}_2(T) = 2$. In cases 3, 4, and 5, $T \cong \mathbb{Z}_2$, implying that $\mathrm{rk}_2(T) = 1$. In the final case, $T \cong \mathbb{Z}_4$, implying that $\mathrm{rk}_2(T) = 1$. We see that for any possible $P \subseteq G_2$ that is a principal 2-radical subgroup of G, $\mathrm{rk}_2(N_G(P)/P) \leq 2$. This concludes the proof of Proposition 5.4. □

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The Ohio State University, 231 W. 18th Ave., Columbus, OH 43210, USA, mjackson@math.ohio-state.edu