The strong symmetric genus of the finite Coxeter groups

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Introduction

Question: Given a finite group G, what is the smallest genus of any closed orientable topological surface on which this group acts faithfully as a group of orientation preserving symmetries?

This least genus is called the strong symmetric genus of the group G and is denoted $\sigma^0(G)$.

By a result of Hurwitz [1893], if $\sigma^0(G) > 1$ for a finite group G, then $\sigma^0(G) \geq 1 + \frac{|G|}{84}$.

Known results on the strong symmetric genus

All groups G such that $\sigma^0(G) \leq 4$ are known.
[Broughton, 1991; May and Zimmerman, 2000 and 2005]

For each positive integer n, there is exists a finite group G with $\sigma^0(G) = n$. [May and Zimmerman, 2003]

The strong symmetric genus is known for the following groups:

- alternating and symmetric groups [Conder, 1980 and 1981]
- $PSL_2(q)$ [Glover and Sjerve, 1985 and 1987]
- $SL_2(q)$ [Voon, 1993]
- the sporadic finite simple groups
  [Conder, Wilson and Woldar, 1992; Wilson, 1993, 1997 and 2001]

Coxeter groups and finite Coxeter groups

Coxeter group = finitely generated Euclidean reflection group. Each has a presentation of the following form :

$$\langle r_1, r_2, \ldots, r_n | (r_i r_j)^{m_{ij}} = 1 \text{ such that } \begin{array}{c} m_{ii} = 1 \text{ and} \\ m_{ij} = m_{ji} \geq 2 \text{ for } i \neq j \end{array} \rangle.$$

The (irreducible) finite Coxeter groups are fully classified and fall into 4 infinite families with 6 other groups.

The finite Coxeter groups are the symmetry groups of regular or semi-regular n-dimensional polytopes.

group of type	Symmetry group (other description)
A_n, $n \geq 1$	regular n-simplex (Symmetric group of:, Σ_{n+1})
$B_n = C_n$, $n \geq 2$	n-dimensional cube or octahedran (hyperoctahedral group, $\mathbb{Z}_2 \wr \Sigma_n$)
D_n, $n \geq 3$	the semiregular polytope whose vertices are half of the vertices of the n-dimensional cube no two of which share an edge in that cube $((\mathbb{Z}_2)^{n-1} \rtimes \Sigma_n)$
I_{2m}, $m \geq 2$	the regular m-gon (Dihedral group of order $2m$)
H_3	the regular icosahedron
H_4	the 4-dim., regular 120-cell (or 600-cell)
F_4	the 4-dim., regular 24-cell
E_6	the E_6 polytope
E_7	the E_7 polytope
E_8	the E_8 polytope

Recall that strong symmetric genus of the symmetric groups is known.

Each finite dihedral group is a group of orientation preserving automorphisms of a sphere

So each finite dihedral group has a strong symmetric genus of 0.

Today we are concerned with the strong symmetric genus of the remaining finite Coxeter groups, especially the infinite families of type B_n and D_n.

The process for finding for finding the strong symmetric genus of a group is to find generators of the group.

We will say that a pair of elements $x, y \in G$, which generate G, is a (p, q, r)-generating pair of G, using the convention that $p \leq q \leq r$, if the orders of x, y, and z are p, q, and r, in some order.

Following the convention of Marston Conder, we say that a (p, q, r)-generating pair of G is a minimal generating pair if there does not exist a (k, l, m)-generating pair for G with $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} > \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$.

Lemma 1 (Singerman) *Let G be a finite group such that $\sigma^0(G) \geq 2$. If G has a (s,t,u)-generating pairs such that $\frac{1}{s}+\frac{1}{t}+\frac{1}{u} \geq \frac{5}{6}$, then for any minimal (p,q,r)-generating pair G,*

$$\sigma^0(G) = 1 + \frac{1}{2}|G| \cdot (1 - \frac{1}{p} - \frac{1}{q} - \frac{1}{r}).$$

The triples which work in Singerman's Lemma:

$(2,3,n)$	$n \geq 7$	$(2,4,n)$	$5 \leq n \leq 12$
$(2,5,n)$	$5 \leq n \leq 7$	$(3,3,n)$	$4 \leq n \leq 5$
$(2,6,6)$			

Some previously known results due to Conder:

- For $n > 47$, the altenating group A_n has a minimal $(2,3,7)$-generating pair.
- For $n > 29$, Σ_n has a minimal $(2,3,8)$-generating pair.

The Hyperoctahedral Groups

• $B_n = \mathbb{Z}_2 \wr \Sigma_n$ and so $|B_n| = n!2^n$.
• Notice that if x and y generate Σ_n for some n, then at most one of x, y, and xy has odd order.

Proposition 2 (J) *If x and y generate B_n, then the orders of x, y, and xy all are even.*

Theorem 3 (J) *For all $n \geq 3$, except n =5, 6, and 8, B_n has a minimal $(2,4,6)$-generating pair. In the exceptional cases, B_5, B_6 and B_8 have minimal generating pairs of type $(2,4,10)$, $(2,6,6)$, and $(2,4,8)$, respectively.*

Corollary 4 (J) *For all $n \geq 3$, except n =5, 6, and 8,* $\sigma^0(B_n) = \frac{n!2^n}{24} + 1 = \frac{n!2^{n-3}}{3} + 1$.

Coxeter groups of type D

• Recall that $D_n = (\mathbb{Z}_2)^{n-1} \rtimes \Sigma_n$ and so $|D_n| = 2^{n-1} n!$.
• For the Coxeter groups of type D we make use of the following two quotients:

$$D_n \cong B_n / Z(B_n) \text{ and } \Sigma_n \cong D_n / (\mathbb{Z}_2)^{n-1}.$$

For almost all n, we show that D_n has a minimal $(2,3,8)$-generating pair by taking a minimal $(2,3,8)$-generating pair of Σ_n and constructing a $(2,3,8)$-generating pair for D_n from it.

Theorem 5 (J) *For each $n \geq 30$, D_n has a minimal $(2,3,8)$-generating pair and $\sigma^0(D_n) = \frac{n!\, 2^{n-1}}{48} = \frac{n!\, 2^{n-5}}{3}$.*

The strong symmetric genus of the 6 individual finite Coxeter groups are given in the following table:

group of type	size	min. gen. pair	$\sigma^0(G)$
H_3	120	$(2,3,10)$	5
H_4	14400	$(2,4,6)$	601
F_4	1152	$(2,6,6)$	97
E_6	$51,840$	$(2,4,8)$	3241
E_7	$2,903,040$	$(2,4,7)$	$155,521$
E_8	$696,729,600$	$(2,4,8)$	$43,545,601$

n	$A_{n-1} = \Sigma_n$	B_n	D_n
3	$\Sigma_3 \cong I_{2(3)}$	$(2,4,6)$	$D_3 \cong \Sigma_4$
4	$(2,3,4)$	$(2,4,6)$	$(3,4,4)*$
5	$(2,4,5)$	$(2,4,10)$	$(2,4,5)$
6	$(2,5,6)$	$(2,6,6)$	$(2,5,6)$
7	$(2,3,10)$	$(2,4,6)$	$(2,4,6)*$
8	$(2,4,7)$	$(2,4,8)$	$(2,4,7)$
9	$(2,4,6)$	$(2,4,6)$	$(2,4,6)$
10	$(2,3,10)$	$(2,4,6)$	$(2,3,10)$
11	$(2,4,5)$	$(2,4,6)$	$(2,4,5)$
12	$(2,3,12)$	$(2,4,6)$	$(2,3,12)$
13	$(2,3,12)$	$(2,4,6)$	$(2,3,12)$
14	$(2,4,6)$	$(2,4,6)$	$(2,3,14)*$
15	$(2,4,5)$	$(2,4,6)$	$(2,4,5)$
16	$(2,4,5)$	$(2,4,6)$	$(2,4,5)$
17	$(2,4,6)$	$(2,4,6)$	$(2,4,6)$
20	$(2,3,8)$	$(2,4,6)$	$(2,4,5)*$
22	$(2,3,10)$	$(2,4,6)$	$(2,3,10)$
23	$(2,3,10)$	$(2,4,6)$	$(2,3,12)*$
26	$(2,4,5)$	$(2,4,6)$	$(2,4,5)$
29	$(2,3,12)$	$(2,4,6)$	$(2,3,12)$