

Algebraic Rules, Properties, Formulas, Ideas and Processes.

1) Rules and Properties of Exponents.

Let x and y be positive real numbers, let a and b represent real numbers, and let n represent a positive integer. Then:

- 1) $x^a x^b = x^{a+b}$
- 2) $\frac{x^a}{x^b} = x^{a-b}$
- 3) $(x^a)^b = x^{a \cdot b}$
- 4) $(x \cdot y)^a = x^a \cdot y^a$
- 5) $\sqrt[n]{x} = x^{1/n}$
- 6) $x^0 = 1$ so long as $x \neq 0$
- 7) $x^{-b} = \frac{1}{x^b}$

The sixth property is a consequence of the second property—check and see what happens if you let $a = b$. Also, the seventh property is a consequence of the second and sixth properties—check and see what happens when $a = 0$.

2) Functions and Composition.

A function $f(x)$ is **even** if $f(-x) = f(x)$. The graph of an even function is symmetric about the y -axis. Examples of even functions include x^2 (or indeed x^n for any even exponent n), $|x|$, and $\cos x$. Both the sum and product of a pair of even functions is also an even function.

A function $f(x)$ is **odd** if $f(-x) = -f(x)$. Note that this definition implies that if $f(x)$ is an odd function, then $f(0) = 0$. The graph of an odd function is symmetric through the origin. Examples of odd functions include x^3 (in fact, x^n for any odd exponent n) and $\sin x$. The sum of a pair of odd functions is also an odd function. The product of an even function and an odd function is an odd function. The product of two odd functions is an even function.

If $f(x)$ and $g(x)$ are a given pair of functions and if the range of $g(x)$ lies in the domain of $f(x)$, then the composition of f with g is the function defined by $(f \circ g)(x) = f(g(x))$. This is a critical and central concept to many branches of mathematics, and is one with which you should be comfortable. It is easy to see that if $e(x) = x$, then $(f \circ e)(x) = (e \circ f)(x) = f(x)$ for any x . In other words, the function $e(x)$ is an identity element with respect to functional composition. (In an analogous manner, the number 0 is the identity with respect to addition and the number 1 is the identity with respect to multiplication.) Two functions $f(x)$ and $g(x)$ are said to be composition inverses of each other (or just inverses) if $(f \circ g)(x) = (g \circ f)(x) = e(x) = x$. In this case, we commonly denote the inverse of $f(x)$ by $f^{-1}(x)$. Verify that if $f(x) = x^3$ then $f^{-1}(x) = \sqrt[3]{x}$ and if $f(x) = 4x + 7$ then $f^{-1}(x) = \frac{x-7}{4}$. Recall that one way to find the inverse of $y = f(x)$ is to write the expression $x = f(y)$ and then to solve for y .

3) Polynomials, Factoring, and Roots.

You should instantly realize that $x^2 - y^2 = (x+y)(x-y)$. This idea lets you rationalize the denominator of expressions such as $\frac{5}{2+\sqrt{7}}$ and $\frac{2}{4-\sqrt{x}}$. You should also be very familiar with factoring polynomials like $x^2 + 6x - 16$ and $6x^2 + 19x + 15$. Factoring a polynomial is, of course, a good way to find its roots. If a polynomial won't factor, then you can always resort to the quadratic formula. If a , b , and c are constants, then the roots of $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. (It turns out that there is also a cubic formula and a quartic formula, but they are grotesquely complicated.) The expression $b^2 - 4ac$ is called the discriminant. If the discriminant is positive, then there are two distinct real roots. If the discriminant is negative, then there are two distinct complex (or imaginary) roots. If the discriminant is 0, then there is a double root at $x = \frac{-b}{2a}$.

An important procedure we will use in the course is completing the square. If $f(x) = x^2 + bx + c$, then we complete the square by adding and subtracting $(\frac{b}{2})^2 = \frac{b^2}{4}$ to $f(x)$. This gives us $f(x) = (x^2 + bx + \frac{b^2}{4}) + c - \frac{b^2}{4} = (x + \frac{b}{2})^2 + c - \frac{b^2}{4}$. You should be able to derive the quadratic formula by dividing both sides of $ax^2 + bx + c = 0$ by a and then completing the square.

While factoring reveals the roots of a polynomial, knowing the roots can let you design a polynomial. For example, if the second degree polynomial $f(x)$ has 3 and -2 for its roots, then $f(x) = a(x+2)(x-3) = a(x^2 - x - 6)$, where some additional piece of information is needed to determine a .

4) A Brief Review of Exponentials and Logarithms

Functions of the type $y = a^x$ where a is a positive constant are called exponential functions. Such functions play a prominent role in calculus in general, and business calculus in particular, particularly when considering interest, either on a principal investment or on a loan. If $a \neq 1$, then the function will be monotonic (always increasing or always decreasing) and thus has an inverse function called a **logarithm**. If we denote this inverse function by $g(x) = \log_a x$, then we have the following two crucial properties:

$$f(g(x)) = a^{\log_a x} = x \quad \text{and} \quad g(f(x)) = \log_a a^x = x$$

Another way to look at it is to realize that $y = \log_a x$ means that $a^y = x$. In particular, the output of a logarithm represents an exponent, and thus logarithms should have properties that mirror the properties of exponents that were mentioned above:

$$\begin{array}{ll} 1) \log_a x + \log_a y = \log_a xy & 2) \log_a x - \log_a y = \log_a \left(\frac{x}{y}\right) \\ 3) b \log_a x = \log_a x^b & 4) \log_a 1 = 0 \end{array}$$

Calculus is an exciting, fun, and challenging subject. I hope that this little review gives you a nice head start and will ease your transition into the course. I am eagerly looking forward to meeting you this spring and wish you all the best of luck in the coming semester.

Dr. Thompson