

Things You Should Know Coming Into Calc II

Algebraic Rules, Properties, Formulas, Ideas and Processes:

1) Rules and Properties of Exponents.

Let x and y be positive real numbers, let a and b represent real numbers, and let n represent a positive integer. Then:

- 1) $x^a x^b = x^{a+b}$ 2) $\frac{x^a}{x^b} = x^{a-b}$
- 3) $(x^a)^b = x^{a \cdot b}$ 4) $(x \cdot y)^a = x^a \cdot y^a$
- 5) $\sqrt[n]{x} = x^{1/n}$ 6) $x^0 = 1$ so long as $x \neq 0$
- 7) $x^{-b} = \frac{1}{x^b}$

The sixth property is a consequence of the second property—check and see what happens if you let $a = b$. Also, the seventh property is a consequence of the second and sixth properties—check and see what happens when $a = 0$.

2) Functions and Composition.

A function $f(x)$ is **even** if $f(-x) = f(x)$. The graph of an even function is symmetric about the y -axis. Examples of even functions include x^2 (or indeed x^n for any even exponent n), $|x|$, and $\cos x$. Both the sum and product of a pair of even functions is also an even function.

A function $f(x)$ is **odd** if $f(-x) = -f(x)$. Note that this definition implies that if $f(x)$ is an odd function, then $f(0) = 0$. The graph of an odd function is symmetric through the origin. Examples of odd functions include x^3 (in fact, x^n for any odd exponent n) and $\sin x$. The sum of a pair of odd functions is also an odd function. The product of an even function and an odd function is an odd function. The product of two odd functions is an even function.

If $f(x)$ and $g(x)$ are a given pair of functions and if the range of $g(x)$ lies in the domain of $f(x)$, then the composition of f with g is the function defined by $(f \circ g)(x) = f(g(x))$. This is a critical and central concept to many branches of mathematics, and is one with which you should be comfortable. It is easy to see that if $e(x) = x$, then $(f \circ e)(x) = (e \circ f)(x) = f(x)$ for any x . In other words, the function $e(x)$ is an identity element with respect to functional composition. (In an analogous manner, the number 0 is the identity with respect to addition and the number 1 is the identity with respect to multiplication.) Two functions $f(x)$ and $g(x)$ are said to be composition inverses of each other (or just inverses) if $(f \circ g)(x) = (g \circ f)(x) = e(x) = x$. In this case, we commonly denote the inverse of $f(x)$ by $f^{-1}(x)$. Verify that if $f(x) = x^3$ then $f^{-1}(x) = \sqrt[3]{x}$ and if $f(x) = 4x + 7$ then $f^{-1}(x) = \frac{x-7}{4}$. Recall that one way to find the inverse of $y = f(x)$ is to write the expression $x = f(y)$ and then to solve for y .

3) Polynomials, Factoring, and Roots.

You should instantly realize that $x^2 - y^2 = (x + y)(x - y)$, $(x + y)^2 = x^2 + 2xy + y^2$, and $(x - y)^2 = x^2 - 2xy + y^2$. The first identity lets you rationalize the denominator of expressions such as $\frac{5}{2+\sqrt{7}}$ and $\frac{2}{4-\sqrt{x}}$. You should also be very familiar with factoring polynomials like $x^2 + 6x - 16$ and $6x^2 + 19x + 15$. Factoring a polynomial is, of course, a good way to find its roots. If a polynomial won't factor, then you can always resort to the quadratic formula. If a , b , and c are constants, then the roots of $ax^2 + bx + c = 0$ are given by $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. (It turns out that there is also a cubic formula and a quartic formula, but

they are grotesquely complicated.) The expression $b^2 - 4ac$ is called the discriminant. If the discriminant is positive, then there are two distinct real roots. If the discriminant is negative, then there are two distinct complex (or imaginary) roots. If the discriminant is 0, then there is a double root at $x = \frac{-b}{2a}$.

An important procedure we will use in the course is completing the square. If $f(x) = x^2 + bx + c$, then we complete the square by adding and subtracting $(\frac{b}{2})^2 = \frac{b^2}{4}$ to $f(x)$. This gives us

$$f(x) = (x^2 + bx + \frac{b^2}{4}) + c - \frac{b^2}{4} = (x + \frac{b}{2})^2 + c - \frac{b^2}{4}.$$

You should be able to derive the quadratic formula by dividing both sides of $ax^2 + bx + c = 0$ by a and then completing the square.

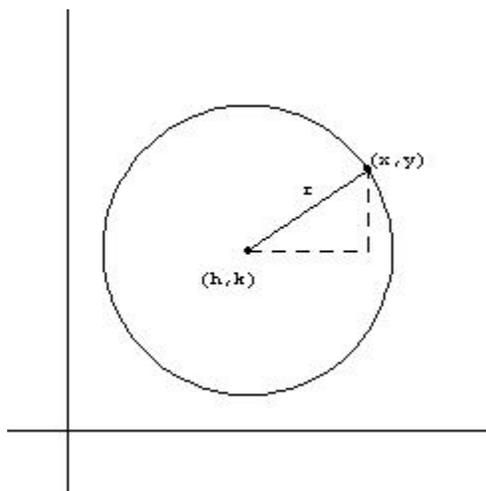
While factoring reveals the roots of a polynomial, knowing the roots can let you design a polynomial. For example, if the second degree polynomial $f(x)$ has 3 and -2 for its roots, then $f(x) = a(x+2)(x-3) = a(x^2 - x - 6)$, where some additional piece of information is needed to determine a .

4) A Brief Review of Conic Sections.

There are 4 conic sections, each defined by second degree polynomials. You should have some idea of what their graphs look like, and how to identify key points and features of their graphs.

A **parabola** is given by the equation $y = ax^2 + bx + c$. The parabola opens up if $a > 0$ and opens down if $a < 0$. By completing the square, you can rewrite the above equation as $y = a(x + \frac{b}{2a})^2 + (c - \frac{b^2}{4a^2})$. The vertex of the parabola is at the point $(-\frac{b}{2a}, c - \frac{b^2}{4a^2})$ (basically, just keep track of the value of x that makes you square 0), and the graph of the parabola is symmetric about the vertical line $x = -\frac{b}{2a}$ that is called, amazingly enough, the axis of symmetry. Note that if $b = 0$, then the function $y = ax^2 + c$ is an even function.

A **circle** is defined to be the set of points that are a fixed distance r from a particular point (h, k) called the center of the circle. The standard form for the equation for a circle is actually nothing more than the Pythagorean Theorem! If you look at the figure, you see that $r^2 = (x - h)^2 + (y - k)^2$ where (h, k) is the center of the circle and (x, y) is some random point on the circle. Note that this example also shows that the formula for the distance between two points is also an application of the Pythagorean Theorem.



An equation of the form $ax^2 + bx + ay^2 + cy = d$ is a nonstandard equation for a circle so long as you end up with a positive number on the right hand side when, after dividing both sides by a , you complete the squares on the left hand side. If the coefficients for x^2 and y^2 are different, then the equation represents either an ellipse or, if the coefficients have opposite signs, a hyperbola. See below.

As mentioned in the above paragraph, the equation $ax^2 + bx + cy^2 + dy = e$ where $a \neq c$ is an ellipse, assuming that a and c are either both positive or both negative. (If a and c have opposite signs, then you have a hyperbola.) This equation may be rewritten as $a(x^2 + \frac{b}{a}x) + c(y^2 + \frac{d}{c}y) = e$. As usual, you can then complete the squares on the left hand side to get $a(x + \frac{b}{2a})^2 + c(y + \frac{d}{2c})^2 = e + \frac{b^2}{4a} + \frac{d^2}{4c}$. Let $E = e + \frac{b^2}{4a} + \frac{d^2}{4c}$. In order to have a real valued expression, E must be > 0 since the left hand side is a sum of two non-negative terms. Assuming a , c , and E are all positive, then so are $\frac{a}{E}$ and $\frac{c}{E}$. Thus both $\frac{E}{a}$ and $\frac{E}{c}$ are positive and could be regarded as being the squares of some pair of numbers, say $\frac{E}{a} = A^2$ and $\frac{E}{c} = B^2$. Thus the original equation for the ellipse can be put into the standard form

$$\frac{(x - h)^2}{A^2} + \frac{(y - k)^2}{B^2} = 1$$

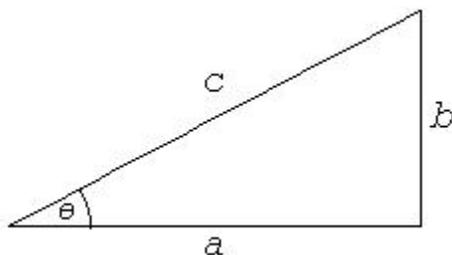
where $h = -\frac{b}{2a}$ and $k = -\frac{d}{2c}$. The center of the ellipse is at (h, k) and the vertices of the ellipse are at $(h \pm A, k)$ and $(h, k \pm B)$.

The equation $ax^2 + bx + cy^2 + dy = e$ represents a **hyperbola** when a and c have opposite signs. Proceeding as we did with the ellipse, the hyperbola can be put into one of two standard forms:

$$\frac{(x - h)^2}{A^2} - \frac{(y - k)^2}{B^2} = 1 \quad \text{or} \quad \frac{(y - k)^2}{B^2} - \frac{(x - h)^2}{A^2} = 1.$$

In the first case the graph of the hyperbola has two branches opening to the right and to the left. In the second case the hyperbola has two branches opening up and down. The center will be at (h, k) in both cases. The first hyperbola has vertices at $(h \pm A, k)$ and the second hyperbola has vertices at $(h, k \pm B)$.

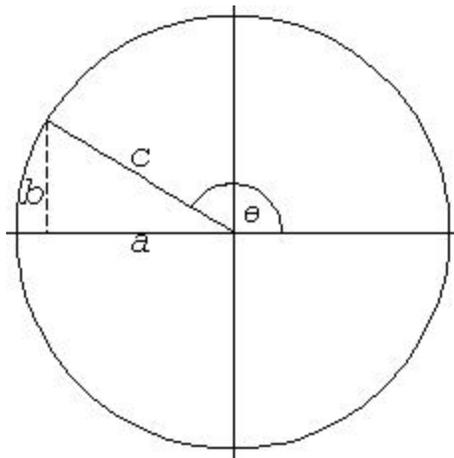
5) A Brief Review of Trigonometry.



Given a right triangle, you should know the definitions of the basic trig functions:

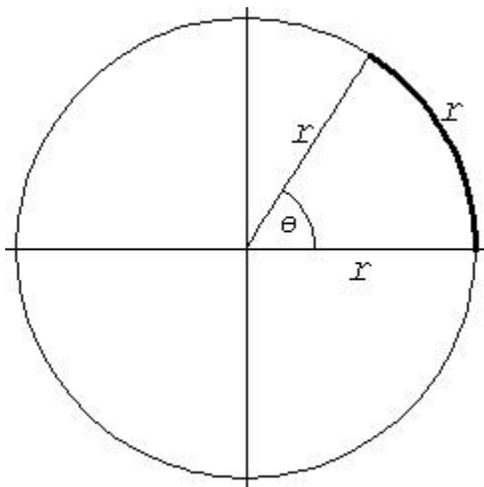
$$\begin{aligned} \sin \theta &= \frac{b}{c} & \csc \theta &= \frac{c}{b} \\ \cos \theta &= \frac{a}{c} & \sec \theta &= \frac{c}{a} \\ \tan \theta &= \frac{b}{a} & \cot \theta &= \frac{a}{b} \end{aligned}$$

The notation $\sin^2 \theta$ means $(\sin \theta)^2$, which is distinct from $\sin(\theta^2)$. Note that by the Pythagorean Theorem, $\sin^2 \theta + \cos^2 \theta = 1$. This in turn implies that $-1 \leq \sin \theta \leq 1$ and $-1 \leq \cos \theta \leq 1$. If we base our definitions of the trig functions exclusively on triangles, then the angle θ is naturally restricted to be between 0 and 90 degrees or, in a calculus setting, between 0 and $\frac{\pi}{2}$ radians. We can extend the definitions of trig functions to arbitrary angles by using a reference triangle inscribed in a circle having its center at the origin, as in the figure below:



In this picture, the angle θ is between $\frac{\pi}{2}$ and π radians (or between 90 and 180 degrees), and so the reference triangle lies in the second quadrant. Note that a is negative since the x -coordinate of any point to the left of the y -axis is negative, and b is positive since the y -coordinate of any point above the x -axis is positive. The hypotenuse is **always** assumed to be positive.

Just as length could be measured in feet or meters, angles of rotation can be measured in degrees or radians. There are several key advantages to using radians, and they all arise from the geometric nature of the definition of a radian. Take a circle of radius r , and then mark off an arc of length r along the perimeter of the circle. (See the figure below.) The angle subtended by that arc is defined to be 1 radian. Since the circumference of a circle is $2\pi r$ and the length of the arc is r , there will be 2π such arcs in the total circumference of the circle. Hence there are 2π radians in a full circle.



Since there are 2π radians in a circle each radian represents $\frac{1}{2\pi}$ th of a circle. Also, the circumference of an entire circle is $2\pi r$. Thus the length of an arc that subtends an angle of θ radians is $2\pi r \cdot \left(\frac{\theta}{2\pi}\right) = r\theta$. In a similar manner, the area of a sector of a circle that subtends an angle of θ radians is $\pi r^2 \cdot \left(\frac{\theta}{2\pi}\right) = \frac{1}{2}r^2\theta$.

SOME USEFUL IDENTITIES:

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

$$\cos^2 \theta = \frac{1}{2}(1 + \cos 2\theta)$$

$$\sin(-\theta) = -\sin \theta$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$\sin^2 \theta = \frac{1}{2}(1 - \cos 2\theta)$$

$$\cos(-\theta) = \cos \theta$$

The last two identities show that $\sin \theta$ is an odd function and $\cos \theta$ is an even function. You should have a good idea of what the graphs of both $\sin \theta$ and $\cos \theta$ look like. Also, you should know that the

periods of $\sin k\theta$ and $\cos k\theta$ are $\frac{2\pi}{k}$, where k is a non-zero constant. In particular, if $k = 1$, then we see that $\sin \theta$ and $\cos \theta$ have a period of 2π .

I expect you to know the values on these tables by heart. Doing so will facilitate many of our discussions and make it easier to design manageable computational examples.

θ	$\sin \theta$	$\cos \theta$
0	0	1
$\pi/6$	1/2	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	1/2
$\pi/2$	1	0

6) Ideas and Results From Calculus

Intuitively, the expression $\lim_{x \rightarrow a} f(x) = L$ means if x gets very close to a , then $f(x)$ gets really close to L . Although we will not be using the technical definition of a limit in this class, it is a notion that every math major will have to eventually assimilate, so here it is:

$$\lim_{x \rightarrow a} f(x) = L \Leftrightarrow (\text{given } \epsilon > 0 \exists \delta \text{ such that } |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon).$$

If that all looks like gibberish to you, that will not be a problem. On the other hand, if it does look familiar, then congratulate your calculus teacher. By the way, δ just measures how close x is to a and ϵ just measures how close $f(x)$ is to L . So the above statement is just a precise way of saying if x gets close to a , then $f(x)$ must be close to L .

Rates of change are typically measured using ratios. Thus we measure velocity by taking distance/time, e.g. miles/hour. Ratios also appear in analytic geometry. In particular, the slope of a line is $\frac{\Delta y}{\Delta x}$. This innocent coincidence is a large part of why calculus exists. The average rate of change of a function $f(x)$ as the independent variable ranges from x to $x + h$ is given by $fsec(x, h) = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$. I am using *fsec* here because geometrically, this expression represents the slope of a secant line, i.e. the slope of a line generated by a pair of points on the graph of $y = f(x)$. In order to measure the **instantaneous** rate of change, we let $h \rightarrow 0$ and get the definition of a derivative:

$$f'(x) = \frac{dy}{dx} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}.$$

Geometrically, the derivative is an expression that allows us to calculate slopes of tangent lines at various points on the graph of $f(x)$. Of course, this expression can be pretty nasty to evaluate, so you learned a bunch of rules and shortcuts to calculate the derivative of various functions:

1) **The Power Rule:** If $f(x) = x^n$, then $f'(x) = nx^{n-1}$. More generally, if $g(x) = (f(x))^n$, then $g'(x) = n(f(x))^{n-1}f'(x)$. This is really a special case of the most important rule of differentiation, the chain rule.

2) **The Product Rule:** $d(f(x)g(x))/dx = f'(x)g(x) + f(x)g'(x)$.

3) **The Quotient Rule:** $d(f(x)/g(x))/dx = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

4) **The Chain Rule:** If $y = f(g(x))$, then $dy/dx = f'(g(x))g'(x)$. This can be reformulated as follows: Let $u = g(x)$ and let $y = f(u)$. Then $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

5) **Trig functions:** $d(\sin x)/dx = \cos x$ $d(\cos x)/dx = -\sin x$. The derivatives of the other trig functions, and of more general expressions involving trig functions, can now be found using the other rules for differentiation **and you should make a point of doing so!**

You should recall that if $f(x)$ has a maximum or a minimum at $x = a$, then $f'(a) = 0$, but **not vice versa**—look at $f(x) = x^3$ at $x = 0$ for example. Also, if $f'(a) > 0$ then $f(x)$ is increasing at $x = a$, and if

$f'(a) < 0$ then $f(x)$ is decreasing at $x = a$. In a similar manner, a function is concave up whenever the second derivative is positive and is concave down whenever the second derivative is negative. Points of inflection occur when the concavity changes, and at these points the second derivative will either be 0 or undefined.

If a derivative represents rate of change, then the antiderivative should represent cumulative change. A natural enough idea that is the complement of differential calculus reviewed above. The fundamental building block of integral calculus is the limit of a Riemann sum:

$$F(b) - F(a) = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(a + i\Delta x)\Delta x, \quad \text{where } \Delta x = \frac{b-a}{n} \text{ and } F'(x) = f(x).$$

Geometrically, we are using the area of a bunch of rectangles to evaluate the area between the graph of $y = f(x)$ and the x -axis (we regard area below the x -axis as “negative” area). But more importantly, we can attach other interpretations to this expression, such as recovering total change of a function given its rate of change, or calculating the volume of a solid, or any of a large number of possibilities. The abstract nature of mathematics allows us to apply this common idea to a wide spectrum of applications, and is one of the triumphs both of mathematics, and of science. Engineers and math majors: You **need** to understand Riemann sums. It is not an idea that you can simply discard after learning about in a superficial manner.

The result that formally ties differential calculus and integral calculus together is known as:

The Fundamental Theorem of Calculus: I) If $F'(x) = f(x)$, then $\int_a^b f(x) dx = F(b) - F(a)$, where $\int_a^b f(x) dx$ represents the limit of the Riemann sum mentioned above.

II) If $F(x) = \int_a^x f(t) dt$, then $F'(x) = f(x)$. More generally, if $F(x) = \int_{g(x)}^{h(x)} f(t) dt$, then $F'(x) = f(h(x)) \cdot h'(x) - f(g(x)) \cdot g'(x)$. This is a direct application of our old friend the chain rule.

The Fundamental Theorem emphasizes the importance of antidifferentiation. There are numerous algorithmic techniques for finding antiderivatives, several of which we will learn in class. You should know how to integrate x^n (so long as $n \neq -1$). You should also know how to integrate $\sin x$, $-\cos x$, $\sec^2 x$, $\sec x \tan x$, $-\csc x \cot x$, and $-\csc^2 x$ since each of these functions is the exact derivative of a trig function. You should also know how to make simple algebraic substitutions to convert messy integrals into simpler integrals. For example, in order to simplify $\int x^2 \cos(x^3) dx$ you would make the substitutions $u = x^3$ and $du = 3x^2 dx$. Note that now $x^2 dx = \frac{1}{3} du$. Thus the above integral is transformed into $\int \frac{1}{3} \cos u du = \frac{1}{3} \sin u + c = \frac{1}{3} \sin(x^3) + c$. Naturally the way to check this answer is to differentiate it. Observe that the x^2 factor appears in the derivative due to the chain rule. Most substitutions in fact are based on taking advantage of the chain rule.

You probably learned that the area between $y = f(x)$ and the x -axis as x ranges from a to b is given by $\int_a^b f(x) dx$. If you take a part of a curve and rotate it about an axis then that curve will “sweep” out a solid. For example, rotating half of a circle will sweep out a sphere. You can now use an integral to calculate the volume of the solid that has been swept out. The generic formula for such a calculation is:

$$V = \int_a^b \pi \rho^2 d_$$

where ρ is the radius of rotation, a and b refer to a range of values for either x or y , and $d_$ will be either dx or dy . If you are rotating about the x -axis, then $\rho = y$ and if you are rotating about the y -axis, then $\rho = x$. The integral represents the (Riemann) sum of the volumes of an infinite number of thin slices (or circular disks). The area of the face of a disk is $\pi \rho^2$ and the thickness of the disk is either dx or dy , depending on the axis of rotation. If you are looking at the region between a pair of curves that are being rotated about

an axis, then the ensuing integral is describing the accumulated volume of an infinite number of washers and looks like:

$$\int_a^b \pi(\rho_1^2 - \rho_2^2) d_-$$

where ρ_1 and ρ_2 are the “big” radius and the “little” radius respectively.

Calculus II is an exciting, fun, and challenging subject. I hope that this little review gives you a nice head start on the course and will ease your transition into the college classroom. I am eagerly looking forward to meeting you and wish you all the best of luck in the coming semester.

Dr. Thompson